# Datenstrukturen und Algorithmen

## Exercise 9

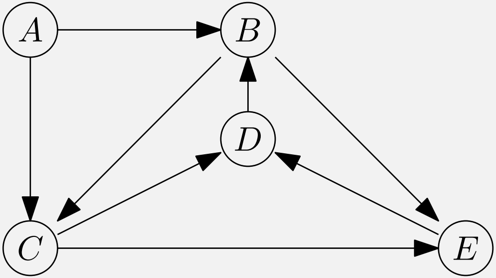
FS 2018

# Program of today

1. Feedback of last exercise
2. Repetition theory
3. Programming Task

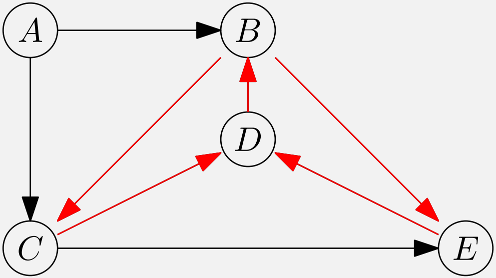
# 1. Feedback of last exercise

# Topological Sorting



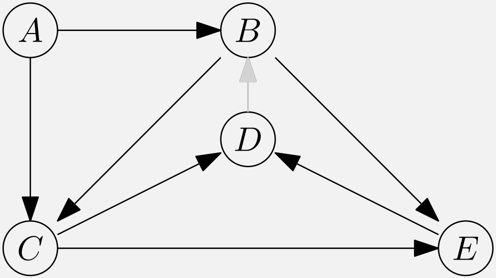

- Graph with cycles

# Topological Sorting



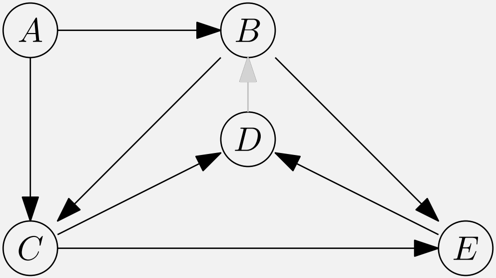

- Graph with cycles
- Two minimal cycles sharing an edge

# Topological Sorting




- Graph with cycles
- Two minimal cycles sharing an edge
- Remove edge $\implies$ cycle-free

# Topological Sorting




- Graph with cycles
- Two minimal cycles sharing an edge
- Remove edge $\implies$ cycle-free
- Topological Sorting by "removing" elements with in-degree 0

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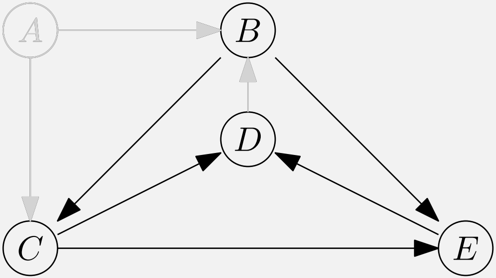


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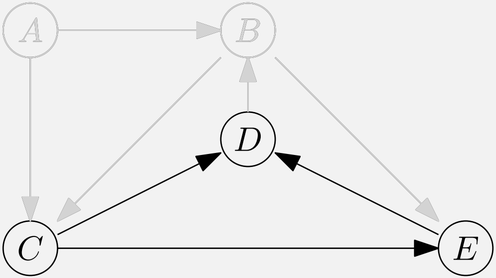


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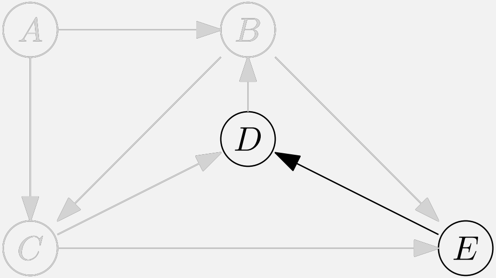
# Topological Sorting

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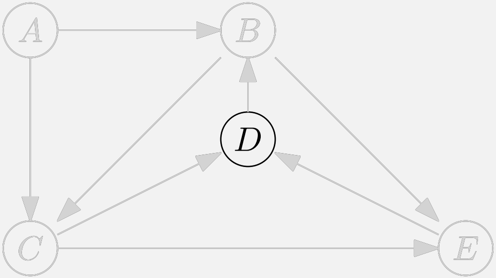
# Topological Sorting

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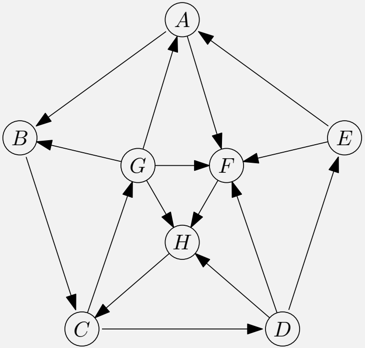
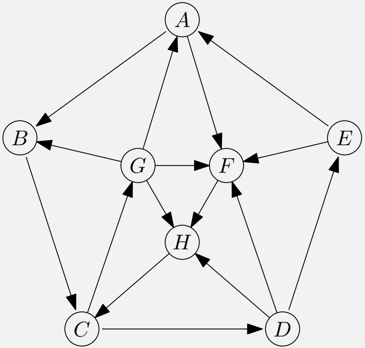
# Depth-first-search and Breadth-first-search




Starting at $A$
DFS: $A, B, C, D, E, F, H, G$
BFS: $A, B, F, C, H, D, G, E$

# Depth-first-search and Breadth-first-search




Starting at $A$

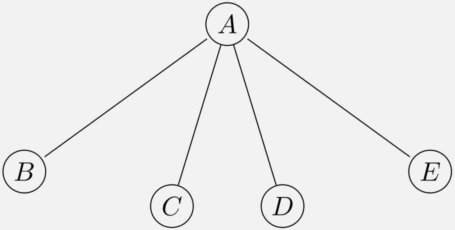
DFS: $A, B, C, D, E, F, H, G$

BFS: $A, B, F, C, H, D, G, E$

There is no starting vertex where the DFS ordering equals the BFS ordering.

# Depth-first-search and Breadth-first-search

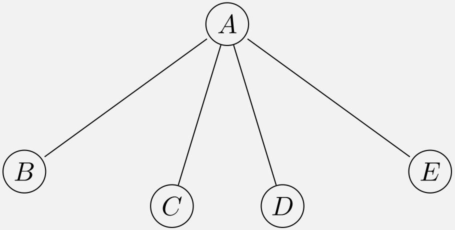
Star: DFS ordering equals BFS ordering




Starting at $A$
DFS: $A, B, C, D, E$
BFS: $A, B, C, D, E$

# Depth-first-search and Breadth-first-search

Star: DFS ordering equals BFS ordering




Starting at $A$
DFS: $A, B, C, D, E$
BFS: $A, B, C, D, E$

Starting at $C$
DFS: $C, A, B, D, E$
BFS: $C, A, B, D, E$

# Huffman Node

```cpp
using SharedNode=std::shared_ptr<Node>;
struct Node{
    char value;
    int frequency;
    SharedNode left;
    SharedNode right;

    // constructor for leafs
    Node(char v, int f): value{v}, frequency{f},
        left{nullptr}, right{nullptr} {}
    // constructor for inner nodes
    Node(SharedNode l, SharedNode r): value{0},
        frequency{l->frequency + r->frequency}, left{l}, right{r} {};
};
```

# Huffman Code- Frequencies

```
std::map<char, int> m;
char x; int n = 0;
while (in.get(x)){
    ++m[x]; ++n;
}
std::cout << "n = " << n << " characters" << std::endl;
```

# Huffman Code - Heap

```
struct comparator {
 bool operator()(const SharedNode a, const SharedNode b) const {
       return a->frequency > b->frequency;
 }
};
       ...

    // build heap
    std::priority_queue<SharedNode, std::vector<SharedNode>, comparato
    for (auto y: m){
       q.push(std::make_shared<Node>(y.first, y.second));
    }
```

# Huffman Code – Tree

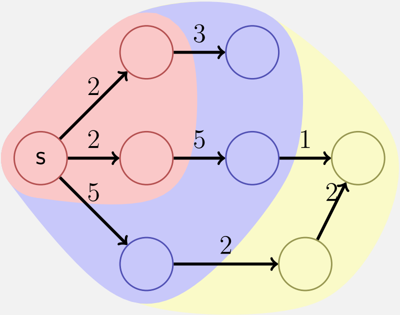
```
// build code tree
SharedNode left;
while (!q.empty()){
    left = q.top();q.pop();
    if (!q.empty()){
        auto right = q.top();q.pop();
        q.push(std::make_shared<Node>(left, right));
    }
}
```

# 2. Repetition theory

# Dijkstra ShortestPath Basic Idea

Set $V$ of nodes is partitioned into

- the set $M$ of nodes for which a shortest path from $s$ is already known,
- the set $R = \bigcup_{v \in M} N^+(v) \setminus M$ of nodes where a shortest path is not yet known but that are accessible directly from $M$,
- the set $U = V \setminus (M \cup R)$ of nodes that have not yet been considered.



# Algorithm Dijkstra

Initial: $\mathrm{PL}(n) \leftarrow \infty$ für alle Knoten.

- Set $\mathrm{PL}(s) \leftarrow 0$
- Start with $M = \{s\}$. Set $k \leftarrow s$.
- While a new node $k$ is added and this is not the target node
  1. For each neighbour node $n$ of $k$:
     - compute path length $x$ to $n$ via $k$
     - If $\mathrm{PL}(n) = \infty$, than add $n$ to $R$
     - If $x < \mathrm{PL}(n) < \infty$, then set $\mathrm{PL}(n) \leftarrow x$ and adapt $R$ .
  2. Choose as new node $k$ the node with smallest path length in $R$.

# General Weighted Graphs

Relaxing Step as with Dijkstra:

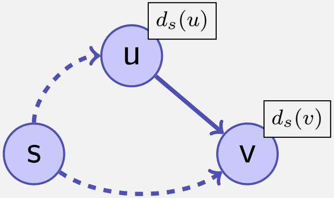
Relax$(u, v)$ $(u, v \in V,\ (u, v) \in E)$

**if** $d_s(v) > d_s(u) + c(u, v)$ **then**

$d_s(v) \leftarrow d_s(u) + c(u, v)$

**return** true

**return** false




Problem: cycles with negative weights can shorten the path, a shortest path is not guaranteed to exist.

# Dynamic Programming Approach (Bellman)

Induction over number of edges $d_s[i,v]$: Shortest path from $s$ to $v$ via maximally $i$ edges.

$$d_s[i,v] = \min\{d_s[i-1,v], \min_{(u,v)\in E}(d_s[i-1,u] + c(u,v))$$

$$d_s[0,s] = 0, d_s[0,v] = \infty \;\forall v \neq s.$$

# DP Induction for all shortest paths

$d^k(u,v) =$ Minimal weight of a path $u \rightsquigarrow v$ with intermediate nodes in $V^k$

Induktion

$$
\begin{aligned}
d^k(u,v) &= \min\{d^{k-1}(u,v), d^{k-1}(u,k) + d^{k-1}(k,v)\}(k \geq 1) \\
d^0(u,v) &= c(u,v)
\end{aligned}
$$

# DP Algorithm Floyd-Warshall($G$)

**Input :** Acyclic Graph $G = (V, E, c)$
**Output :** Minimal weights of all paths $d$

$d^0 \leftarrow c$
**for** $k \leftarrow 1$ **to** $|V|$ **do**
    **for** $i \leftarrow 1$ **to** $|V|$ **do**
        **for** $j \leftarrow 1$ **to** $|V|$ **do**
            $d^k(v_i, v_j) = \min\{d^{k-1}(v_i, v_j), d^{k-1}(v_i, v_k) + d^{k-1}(v_k, v_j)\}$

Runtime: $\Theta(|V|^3)$

Remark: Algorithm can be executed with a single matrix $d$ (in place).

# Algorithm Johnson($G$)

**Input :** Weighted Graph $G = (V, E, c)$
**Output :** Minimal weights of all paths $D$.

New node $s$. Compute $G' = (V', E', c')$
**if** BellmanFord($G', s$) $=$ false **then** return "graph has negative cycles"
**foreach** $v \in V'$ **do**
    $h(v) \leftarrow d(s, v)$ // $d$ aus BellmanFord Algorithmus

**foreach** $(u, v) \in E'$ **do**
    $\tilde{c}(u, v) \leftarrow c(u, v) + h(u) - h(v)$

**foreach** $u \in V$ **do**
    $\tilde{d}(u, \cdot) \leftarrow$ Dijkstra($\tilde{G}', u$)
    **foreach** $v \in V$ **do**
        $D(u, v) \leftarrow \tilde{d}(u, v) + h(v) - h(u)$

# Comparison of the approaches

| Algorithm | | | Runtime |
|---|---|---|---|
| Dijkstra (Heap) | $c_v \geq 0$ | 1:n | $\mathcal{O}(\lvert E\rvert \log \lvert V\rvert)$ |
| Dijkstra (Fibonacci-Heap) | $c_v \geq 0$ | 1:n | $\mathcal{O}(\lvert E\rvert + \lvert V\rvert \log \lvert V\rvert)$ * |
| Bellman-Ford | | 1:n | $\mathcal{O}(\lvert E\rvert \cdot \lvert V\rvert)$ |
| Floyd-Warshall | | n:n | $\Theta(\lvert V\rvert^3)$ |
| Johnson | | n:n | $\mathcal{O}(\lvert V\rvert \cdot \lvert E\rvert \cdot \log \lvert V\rvert)$ |
| Johnson (Fibonacci-Heap) | | n:n | $\mathcal{O}(\lvert V\rvert^2 \log \lvert V\rvert + \lvert V\rvert \cdot \lvert E\rvert)$ * |

* amortized

# Algorithm MST-Kruskal($G$)

**Input :** Weighted Graph $G = (V, E, c)$
**Output :** Minimum spanning tree with edges $A$.

Sort edges by weight $c(e_1) \leq ... \leq c(e_m)$
$A \leftarrow \emptyset$
**for** $k = 1$ **to** $m$ **do**
    **if** $(V, A \cup \{e_k\})$ acyclic **then**
        $A \leftarrow E' \cup \{e_k\}$

**return** $(V, A, c)$

# 3. Programming Task

# Closeness Centrality

- Given: an adjacency matrix for an *undirected* graph on $n$ vertices.
- Output: the *closeness centrality* $C(v)$ of every vertex $v$.

$$C(v) = \sum_{u \in V \setminus \{v\}} d(v, u)$$

# Closeness Centrality

- Given: an adjacency matrix for an *undirected* graph on $n$ vertices.
- Output: the *closeness centrality* $C(v)$ of every vertex $v$.

$$C(v) = \sum_{u \in V \setminus \{v\}} d(v, u)$$

- Intuition: If many connected vertices are close to $v$, then $C(v)$ is small.
- "How central is the vertex in its connected component?"

# All Pairs Shortest Paths

- We require $d(u, v)$ for all vertex pairs $(u, v)$.
- $\implies$ compute all shortest paths using Floyd-Warshall.

```
template<typename Matrix>
void allPairsShortestPaths(unsigned n, Matrix& m)
{
  // your code here
```

- Simply overwrite `m` with the distance values.
- Attention: initially `0` means "no edge".
- Undirected graph: `m[i][j] == m[j][i]`

# Closeness Centrality

```
vector<vector<unsigned> > adjacencies(n,
                  vector<unsigned>(n, 0));
vector<string> names(n);
// ...
allPairsShortestPaths(n, adjacencies);
for(unsigned i = 0; i < n; ++i) {
  cout << names[i] << ": ";
  unsigned centrality = 0;
  // your code here
  cout << centrality << endl;
}
```

# Closeness Centrality: Input Data

- A graph that stems from collaborations on scientific papers.
- The vertices of the graph are the co-authors of the mathematician Paul Erdős.
- There is an edge between them if the authors have jointly published a paper.
- Source: `https://oakland.edu/enp/thedata/`

# Closeness Centrality: Output

```
vertices: 511
ABBOTT, HARVEY LESLIE                   : 1625
ACZEL, JANOS D.                         : 1681
AGOH, TAKASHI                           : 2132
AHARONI, RON                            : 1578
AIGNER, MARTIN S.                       : 1589
AJTAI, MIKLOS                           : 1492
ALAOGLU, LEONIDAS*                      : 0
ALAVI, YOUSEF                           : 1561
...
```

Where does the 0 come from?

Questions?