

Datenstrukturen und Algorithmen

Exercise 5

FS 2018

Program of today

1 Feedback of last exercise

2 Repetition theory

3 Programming Task

Amortized analysis: push_back

Strategy: double if array is full.

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Let $i \in \mathbb{N}$ be the number of elements appended and let $n_i \in \mathbb{N}$ be the array size allocated after appending i .

It holds that

$$n_i = \begin{cases} 1 & \text{if } i = 1 \text{ [Start]} \\ 2 \cdot n_{i-1} & \text{if } i - 1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ n_{i-1} & \text{otherwise} \end{cases}$$

i	n_i
1	1
2	2
3	4
4	4
5	8
6	8
..	..

$$n_i = 2^{\lceil \log_2 i \rceil}$$

Amortized analysis: push_back

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¹According to the task description: $2n$ initialisations, n copies, 1 new element

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Real costs

$$t_i = \begin{cases} 1 & \text{if } i = 1 \text{ [Start]} \\ 3n_{i-1} + 1 & \text{if } i - 1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]}^1 \\ 1 & \text{otherwise} \end{cases}$$

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Find potential function such that the amortized costs are constant:

$$a_i = t_i + \Phi_i - \Phi_{i-1}$$

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$$a_i = t_i + \Phi_i - \Phi_{i-1}$$

$\Phi_i = 6 \cdot$ number of elements in the upper half of the array

$$= 6 \cdot \left(i - \frac{n_i}{2}\right) = 6i - 3n_i$$

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$$\Phi_i - \Phi_{i-1} = \begin{cases} 6 + 3n_{i-1} - 3 \overbrace{n_i}^{2 \cdot n_{i-1}} & \text{if } i-1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ 6 & \text{otherwise} \end{cases}$$

Amortized analysis: push_back

Strategy: double if array is full.

Find potential function such that the amortized costs are constant:

$$\begin{aligned} a_i &= t_i + \Phi_i - \Phi_{i-1} \\ &= \begin{cases} 3n_{i-1} + 1 + 6 - 3n_{i-1} & \text{if } i-1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ 1 + 6 & \text{otherwise} \end{cases} \\ &\leq 7 \quad \text{for all } i \end{aligned}$$

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Strategy: halve if array is three quarters empty.

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Find potential function such that the amortized costs are constant:

$$a_i = t_i + \Phi_i - \Phi_{i-1}$$

Let k_i be the number of elements in the array in step i

$\Phi_i = 3 \cdot \text{number of empty elements in the lower half of the array } (1, \dots, \frac{n}{2})$

$$= 3 \cdot \left(\frac{n_i}{2} - k_i \right)$$

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$$\Rightarrow 4 \geq a_i \text{ (in both cases)}$$

Amortized analysis: pop and push

$$\begin{aligned}\Phi_i = & 6 \cdot \text{number elements in the upper half} \\ & + 3 \cdot \text{number empty slots in the lower half}\end{aligned}$$

2. Repetition theory

Examples of Good Hash Functions

- $h(k) = k \bmod m$, m prime
- $h(k) = \lfloor m(k \cdot r - \lfloor k \cdot r \rfloor) \rfloor$, r irrational, particularly good:
 $r = \frac{\sqrt{5}-1}{2}$.

Open Addressing

Store the colliding entries directly in the hash table using a *probing function* $s(j, k)$ ($0 \leq j < m$, $k \in \mathcal{K}$)

Key table position along a *probing sequence*

$$S(k) := (h(k) + s(0, k), \dots, h(k) + s(m - 1, k)) \text{ mod } m$$

Algorithms for open addressing

- **search(k)** Traverse table entries according to $S(k)$. If k is found, return true. If the probing sequence is finished or an empty position is reached, return false.
- **insert(k)** Search for k in the table according to $S(k)$. If k is not present, insert k at the first free position in the probing sequence.²
- **delete(k)** Search k in the table according to $S(k)$. If k is found, mark the position of k with a **deleted** flag

²A position is also free when it is non-empty and contains a **deleted** flag.

Linear Probing

$$s(j, k) = j \Rightarrow$$

$$S(k) = (h(k) \bmod m, (h(k) + 1) \bmod m, \dots, (h(k) - 1) \bmod m)$$

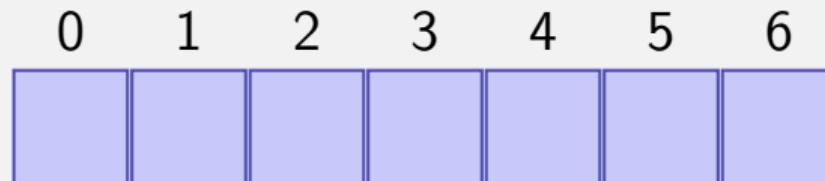
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Example $m = 7$, $\mathcal{K} = \{0, \dots, 500\}$, $h(k) = k \bmod m$.

Key 12



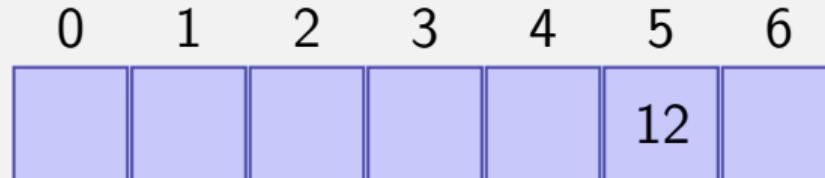
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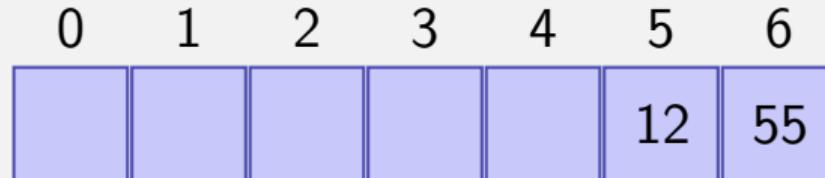
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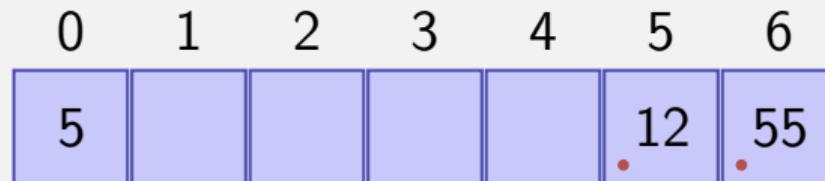
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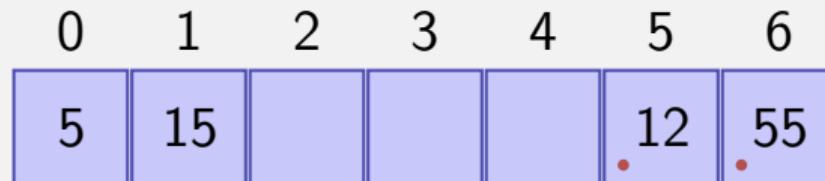
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$$s(j, k) = \lceil j/2 \rceil^2 (-1)^{j+1}$$

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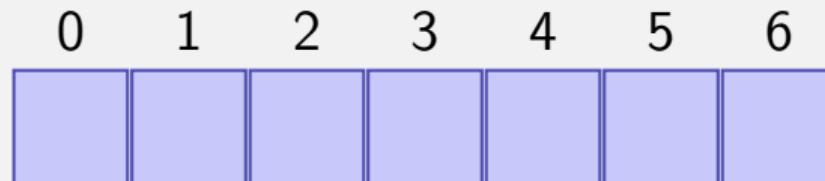
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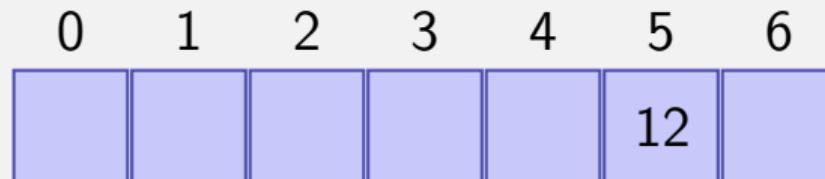
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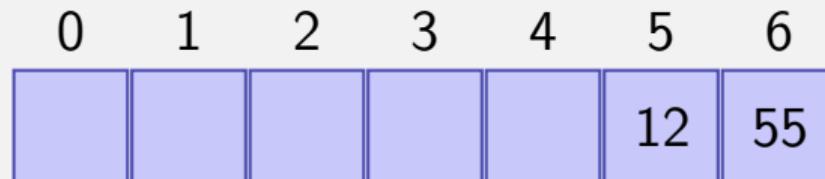
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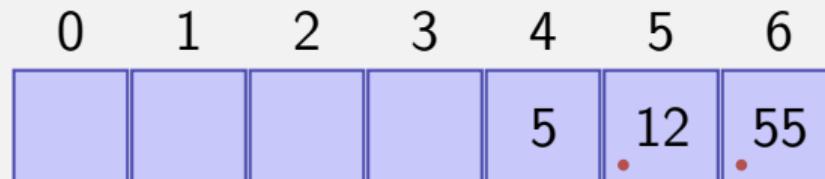
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Double Hashing

Two hash functions $h(k)$ and $h'(k)$. $s(j, k) = j \cdot h'(k)$.

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$$m = 7, \mathcal{K} = \{0, \dots, 500\}, h(k) = k \text{ mod } 7, h'(k) = 1 + k \text{ mod } 5.$$

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3. Programming Task

Finding a Sub-Array

- Given: two integer arrays $A = (a_0, \dots, a_{n-1})$ and $B = (b_0, \dots, b_{k-1})$
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 - $O(nk)$ comparison operations
- Solution using hashing: Calculate hash $h(B)$ and compare it to $h((a_i, a_{i+1}, \dots, a_{i+k-1}))$.
- Avoid re-computing $h((a_i, a_{i+1}, \dots, a_{i+k-1}))$ for each i
 $\implies O(n)$ expected

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 - Can be updated easily: subtract a_i and add a_{i+k} .
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- Better:

$$H_{c,m}((a_i, \dots, a_{i+k-1})) = \left(\sum_{j=0}^{k-1} a_{i+j} \cdot c^{k-j-1} \right) \bmod m$$

- $c = 1021$ prime number
- $m = 2^{15}$ `int`, no overflows at calculations

Sliding Window Hash

```
template<typename It1, typename It2>
It1 findOccurrence(const It1 from, const It1 to,
                    const It2 begin, const It2 end)
{
    const unsigned k = end - begin;
    const unsigned M = 32768;
    const unsigned C = 1021;

    // your code here
    // ...
```

Sliding Window Hash

```
// elements can be compared using std::equal:  
if(std::equal(window_left, window_right, begin, end))  
    return current;  
  
// if no occurrence is found return end of array  
return to;  
}
```

Sliding Window Hash

Make sure that

- the algorithm computes c^k only once,
- all computations are modulo m for all values in order not to get an overflow (recall the rules of modular arithmetic), and
- the values are always positive (e.g., by adding multiples of m).

Questions?

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Let's get to work.