8. Sorting II

Heapsort, Quicksort, Mergesort

8.1 Heapsort

[Ottman/Widmayer, Kap. 2.3, Cormen et al, Kap. 6]

Heapsort

Inspiration from selectsort: fast insertion

Inspiration from insertion sort: fast determination of position

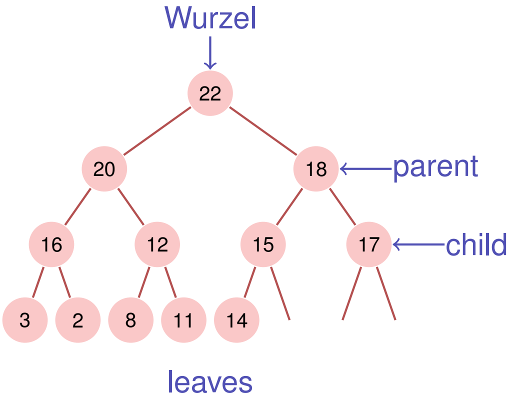
? Can we have the best of two worlds?

! Yes, but it requires some more thinking...

[Max-]Heap[6]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right
3. *Heap-Condition:* Max-(Min-)Heap: key of a child smaller (greater) thant that of the parent node

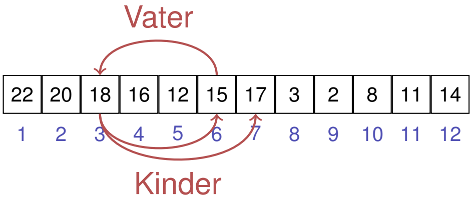


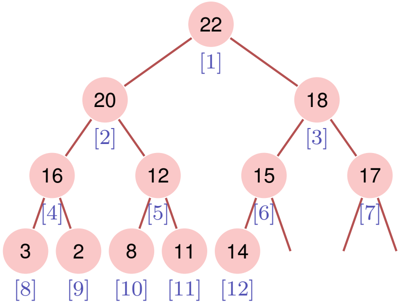
[6] Heap(data structure), not: as in "heap and stack" (memory allocation)

Heap and Array

Tree $\rightarrow$ Array:

- children$(i) = \{2i, 2i+1\}$
- parent$(i) = \lfloor i/2 \rfloor$



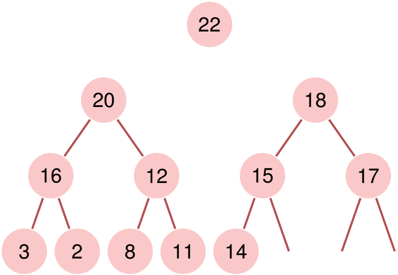


Depends on the starting index[7]

[7]For array that start at 0: $\{2i, 2i+1\} \rightarrow \{2i+1, 2i+2\}$, $\lfloor i/2 \rfloor \rightarrow \lfloor (i-1)/2 \rfloor$

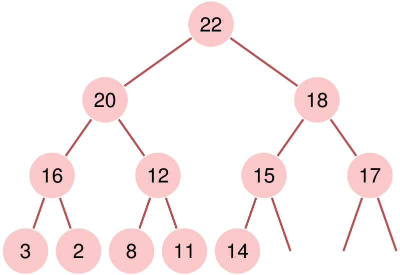
Recursive heap structure

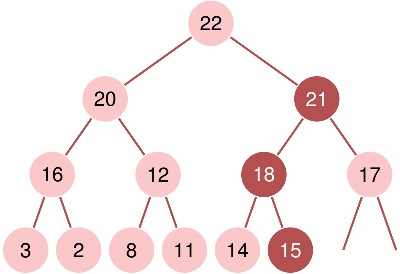
A heap consists of two heaps:



Insert

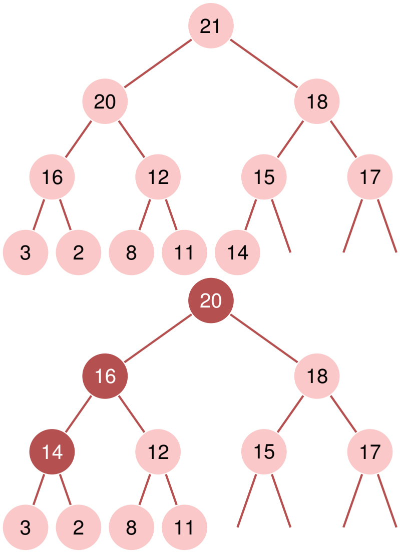
- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively
- Worst case number of operations: $\mathcal{O}(\log n)$





Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)
- Worst case number of operations: $\mathcal{O}(\log n)$



Algorithm Sink(A, i, m)

Input : Array A with heap structure for the children of i. Last element m.
Output : Array A with heap structure for i with last element m.

```
while 2i ≤ m do
    j ← 2i; // j left child
    if j < m and A[j] < A[j+1] then
        j ← j + 1; // j right child with greater key
    if A[i] < A[j] then
        swap(A[i], A[j])
        i ← j; // keep sinking
    else
        i ← m; // sinking finished
```

Sort heap

$A[1,...,n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- Sink($A, 1, n-1$);
- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
sink	$\Rightarrow$	6	5	4	2	1	7
swap	$\Rightarrow$	1	5	4	2	6	7
sink	$\Rightarrow$	5	4	2	1	6	7
swap	$\Rightarrow$	1	4	2	5	6	7
sink	$\Rightarrow$	4	1	2	5	6	7
swap	$\Rightarrow$	2	1	4	5	6	7
sink	$\Rightarrow$	2	1	4	5	6	7
swap	$\Rightarrow$	1	2	4	5	6	7

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence: Induction from below!

Algorithm HeapSort(A, n)

Input : Array A with length n.
Output : A sorted.
for $i \leftarrow n/2$ **downto** 1 **do**
    Sink(A, i, n);
// Now A is a heap.
for $i \leftarrow n$ **downto** 2 **do**
    swap($A[1], A[i]$)
    Sink($A, 1, i - 1$)
// Now A is sorted.

Analysis: sorting a heap

Sink traverses at most $\log n$ nodes. For each node 2 key comparisons. $\Rightarrow$ sorting a heap costs is the worst case $2 \log n$ comparisons.

Number of memory movements of sorting a heap also $\mathcal{O}(n \log n)$.

Analysis: creating a heap

Calls to sink: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

But mean length of sinking paths is much smaller:

$$v(n) = \sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot c \cdot h \in \mathcal{O}(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h})$$

$s(x) := \sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$ $(0 < x < 1)$. With $s(\frac{1}{2}) = 2$:

$$v(n) \in \mathcal{O}(n).$$

8.2 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

? Disadvantages of heapsort?

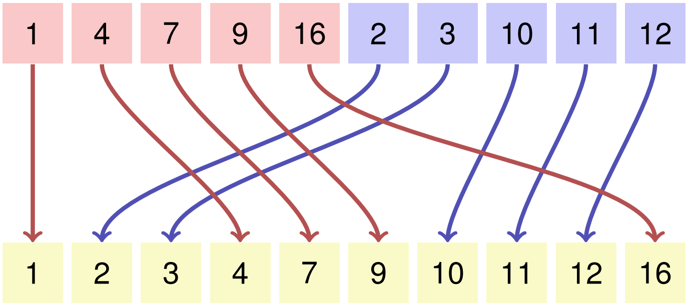
- ! Missing locality: heapsort jumps around in the sorted array (negative cache effect).
- ! Two comparisons before each required memory movement.

Mergesort

Divide and Conquer!

- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with two comparisons.
- Iteratively: sort the presorted array A in $\mathcal{O}(n)$.

Merge



Algorithm Merge(A, l, m, r)

Input : Array A with length n, indexes $1 \leq l \leq m \leq r \leq n$. $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted

Output : $A[l, \dots, r]$ sortiert

$B \leftarrow$ new Array$(r - l + 1)$

$i \leftarrow l$; $j \leftarrow m + 1$; $k \leftarrow 1$

while $i \leq m$ and $j \leq r$ **do**

  if $A[i] \leq A[j]$ **then** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$

  else $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$

  $k \leftarrow k + 1$;

while $i \leq m$ **do** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$; $k \leftarrow k + 1$

while $j \leq r$ **do** $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$; $k \leftarrow k + 1$

for $k \leftarrow l$ **to** r **do** $A[k] \leftarrow B[k - l + 1]$

Correctness

Hypothesis: after k iterations of the loop in line 3 $B[1, \ldots, k]$ is sorted and $B[k] \leq A[i]$, if $i \leq m$ and $B[k] \leq A[j]$ falls $j \leq r$.

Proof by induction:
Base clause: the empty array $B[1, \ldots, 0]$ is trivially sorted.
Induction step ($k \to k+1$):

- wlog $A[i] \leq A[j]$, $i \leq m, j \leq r$.
- $B[1, \ldots, k]$ is sorted by hypothesis and $B[k] \leq A[i]$.
- After $B[k+1] \leftarrow A[i]$ $B[1, \ldots, k+1]$ is sorted.
- $B[k+1] = A[i] \leq A[i+1]$ (if $i+1 \leq m$) and $B[k+1] \leq A[j]$ if $j \leq r$.
- $k \leftarrow k+1, i \leftarrow i+1$: Statement holds again.

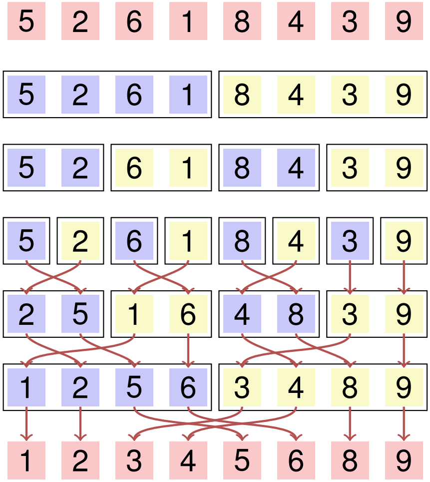
Analysis (Merge)

Lemma

If: array A with length n, indexes $1 \leq l < r \leq n$. $m = \lfloor (l + r)/2 \rfloor$ and $A[l, \dots, m]$, $A[m + 1, \dots, r]$ sorted.
Then: in the call of Merge(A, l, m, r) a number of $\Theta(r - l)$ key movements and comparison are executed.

Proof: straightforward(Inspect the algorithm and count the operations.)

Mergesort



Split

Split

Split

Merge

Merge

Merge

Algorithm recursive 2-way Mergesort(A, l, r)

Input : Array A with length n. $1 \leq l \leq r \leq n$
Output : Array $A[l, \ldots, r]$ sorted.

if $l < r$ **then**
    $m \leftarrow \lfloor (l + r)/2 \rfloor$ // middle position
    Mergesort(A, l, m) // sort lower half
    Mergesort($A, m + 1, r$) // sort higher half
    Merge(A, l, m, r) // Merge subsequences

Analysis

Recursion equation for the number of comparisons and key movements:

$$C(n) = C(\left\lceil \frac{n}{2} \right\rceil) + C(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n) \in \Theta(n \log n)$$

Algorithm StraightMergesort(A)

Avoid recursion: merge sequences of length $1, 2, 4, \ldots$ directly

Input : Array A with length n
Output : Array A sorted

```
length ← 1
while length < n do                        // Iteriere über die Längen n
    right ← 0
    while right + length < n do            // Iteriere über die Teilfolgen
        left ← right + 1
        middle ← left + length − 1
        right ← min(middle + length, n)
        Merge(A, left, middle, right)
    length ← length · 2
```

Analysis

Like the recursive variant, the straight 2-way mergesort always executed a numbe rof $\Theta(n \log n)$ key comparisons and key movements.

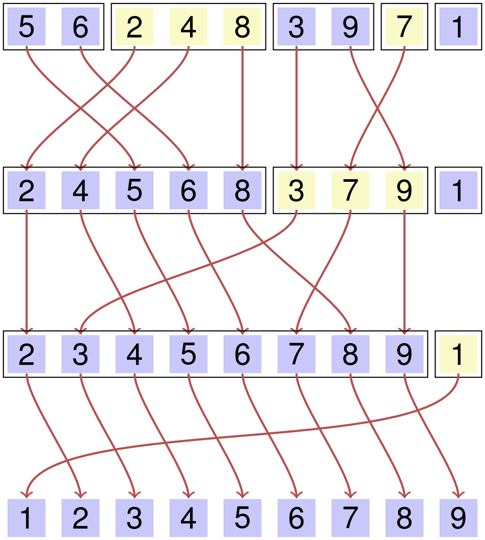
Natural 2-way mergesort

Obserbation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

? How can partially presorted arrays be sorted better?

! Recursive merging of previously sorted parts (*runs) of* A.

Natural 2-way mergesort



Algorithm NaturalMergesort(A)

Input : Array A with length $n > 0$
Output : Array A sorted

```
repeat
    r ← 0
    while r < n do
        l ← r + 1
        m ← l; while m < n and A[m + 1] ≥ A[m] do m ← m + 1
        if m < n then
            r ← m + 1; while r < n and A[r + 1] ≥ A[r] do r ← r + 1
            Merge(A, l, m, r);
        else
            r ← n
until l = 1
```

Analysis

In the best case, natural merge sort requires only $n - 1$ comparisons.

? Is it also asymptotically better than StraightMergesort on average?

! No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions i with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

8.3 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

Quicksort

? What is the disadvantage of Mergesort?

! Requires $\Theta(n)$ storage for merging.

? How could we reduce the merge costs?

! Make sure that the left part contains only smaller elements than the right part.

? How?

! Pivot and Partition!

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Algorithm Quicksort($A[l, \dots, r]$

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.

if $l < r$ **then**
    Choose pivot $p \in A[l, \dots, r]$
    $k \leftarrow$ Partition$(A[l, \dots, r], p)$
    Quicksort$(A[l, \dots, k-1])$
    Quicksort$(A[k+1, \dots, r])$

Reminder: algorithm Partition($A[l, \ldots, r], p$)

Input : Array A, that contains the sentinel p in $[l, r]$ at least once.
Output : Array A partitioned around p. Returns the position of p.

```
while l < r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then                 // Only for keys that are not pairwise different
        l ← l + 1
return l-1
```

Analysis: number comparisons

Best case. Pivot = median; number comparisons:

$$T(n) = 2T(n/2) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \mathcal{O}(n \log n)$$

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

! 2. The maximum number of swaps is given by the number of keys in the smaller part.

Analysis: number swaps

Intellectual game

- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less or equal than half the previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Consequence: there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely if the choice of the pivot and the presorting is not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.

Analysis (randomized quicksort)

Expected number of compared keys with input length n:

$$T(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right),\ T(0) = T(1) = 0$$

Claim $T(n) \leq 4n\log n$.

Proof by induction:

Base clause straightforward for $n = 0$ (with $0\log 0 := 0$) and for $n = 1$.

Hypothesis: $T(n) \leq 4n\log n$ für ein n.

Induction step: $(n-1 \to n)$

Analysis (randomized quicksort)

$$
\begin{aligned}
T(n) &= n-1+\frac{2}{n}\sum_{k=0}^{n-1}T(k) \overset{\text{H}}{\leq} n-1+\frac{2}{n}\sum_{k=0}^{n-1}4k\log k \\
&= n-1+\sum_{k=1}^{n/2}4k\underbrace{\log k}_{\leq \log n-1}+\sum_{k=n/2+1}^{n-1}4k\underbrace{\log k}_{\leq \log n} \\
&\leq n-1+\frac{8}{n}\left((\log n-1)\sum_{k=1}^{n/2}k+\log n\sum_{k=n/2+1}^{n-1}k\right) \\
&= n-1+\frac{8}{n}\left((\log n)\cdot\frac{n(n-1)}{2}-\frac{n}{4}\left(\frac{n}{2}+1\right)\right) \\
&= 4n\log n-4\log n-3\leq 4n\log n
\end{aligned}
$$

Analysis (randomized quicksort)

Theorem

On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ *comparisons.*

Practical considerations

Worst case recursion depth $n - 1$[8]. The also memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

[8]stack overflow possible!

Quicksort with logarithmic memory consumption

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.

while $l < r$ **do**
    Choose pivot $p \in A[l, \ldots, r]$
    $k \leftarrow$ Partition$(A[l, \ldots, r], p)$
    if $k - l < r - k$ **then**
        Quicksort$(A[l, \ldots, k-1])$
        $l \leftarrow k + 1$
    else
        Quicksort$(A[k+1, \ldots, r])$
        $r \leftarrow k - 1$

The call of Quicksort($A[l, \ldots, r]$) in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.

Practical considerations.

Practically the pivot is often the median of three elements. For example: Median3($A[l], A[r], A[\lfloor l + r/2 \rfloor]$).

There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.