20. Dynamic Programming II

Subset sum problem, knapsack problem, greedy algorithm, solutions with dynamic programming, FPTAS, Optimal Search Tree [Ottman/Widmayer, Kap. 7.2, 7.3, 5.7, Cormen et al, Kap. 15,35.5]

Task

Hannes and Niklas shall get a significant amount of presents with different monetary value.

The parents want to distribute the presents in a fair way such that no conflict arises.

Answer: people with children know that there is no solution to this task.

More Realistic Task

Partition the set of the “item” above into two set such that both sets have the same value.

A solution:

Subset Sum Problem

Consider $n \in \mathbb{N}$ numbers $a_1, \ldots, a_n \in \mathbb{N}$.

Goal: decide if a selection $I \subseteq \{1, \ldots, n\}$ exists such that

$$\sum_{i \in I} a_i = \sum_{i \in \{1,\ldots,n\} \setminus I} a_i.$$

Naive Algorithm

Check for each bit vector $b = (b_1, \dots, b_n) \in \{0,1\}^n$, if

$$\sum_{i=1}^{n} b_i a_i \stackrel{?}{=} \sum_{i=1}^{n} (1-b_i) a_i$$

Worst case: n steps for each of the 2^n bit vectors b. Number of steps: $\mathcal{O}(n \cdot 2^n)$.

Algorithm with Partition

- Partition the input into two equally sized parts $a_1, \dots, a_{n/2}$ and $a_{n/2+1}, \dots, a_n$.
- Iterate over all subsets of the two parts and compute partial sum $S_1^k, \dots, S_{2^{n/2}}^k$ ($k = 1, 2$).
- Sort the partial sums: $S_1^k \le S_2^k \le \dots \le S_{2^{n/2}}^k$.
- Check if there are partial sums such that $S_i^1 + S_j^2 = \frac{1}{2}\sum_{i=1}^n a_i =: h$
  - Start with $i = 1, j = 2^{n/2}$.
  - If $S_i^1 + S_j^2 = h$ then finished
  - If $S_i^1 + S_j^2 > h$ then $j \leftarrow j - 1$
  - If $S_i^1 + S_j^2 < h$ then $i \leftarrow i + 1$

Example

Set $\{1,6,2,3,4\}$ with value sum 16 has 32 subsets.

Partitioning into $\{1,6\}$, $\{2,3,4\}$ yields the following 12 subsets with value sums:

$\{1,6\}$				$\{2,3,4\}$							
$\{\}$	$\{1\}$	$\{6\}$	$\{1,6\}$	$\{\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{2,3,4\}$
0	$\boxed{1}$	6	7	0	2	3	4	5	6	$\boxed{7}$	9

$\Leftrightarrow$ One possible solution: $\{1,3,4\}$

Analysis

- Generate partial sums for each part: $\mathcal{O}(2^{n/2} \cdot n)$.
- Each sorting: $\mathcal{O}(2^{n/2} \log(2^{n/2})) = \mathcal{O}(n 2^{n/2})$.
- Merge: $\mathcal{O}(2^{n/2})$

Overal running time

$$\mathcal{O}\left(n \cdot 2^{n/2}\right) = \mathcal{O}\left(n \left(\sqrt{2}\right)^n\right).$$

Substantial improvement over the naive method – but still exponential!

Dynamic programming

Task: let $z = \frac{1}{2}\sum_{i=1}^{n} a_i$. Find a selection $I \subset \{1, \dots, n\}$, such that $\sum_{i \in I} a_i = z$.

DP-table: $[0, \dots, n] \times [0, \dots, z]$-table T with boolean entries. $T[k, s]$ specifies if there is a selection $I_k \subset \{1, \dots, k\}$ such that $\sum_{i \in I_k} a_i = s$.

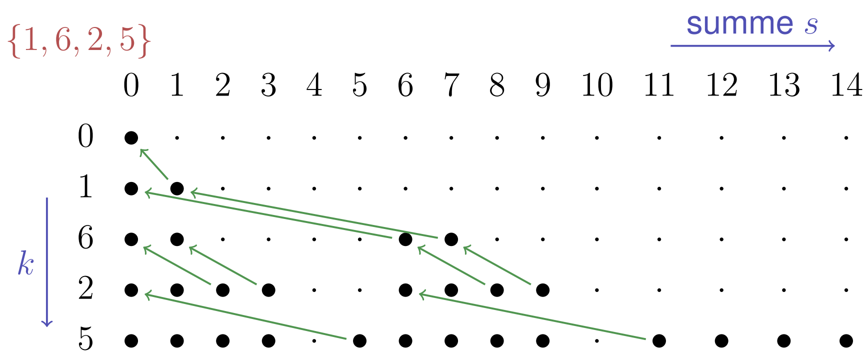
Initialization: $T[0, 0] =$ true. $T[0, s] =$ false for $s > 1$.

Computation:

$$T[k, s] \leftarrow \begin{cases} T[k-1, s] & \text{if } s < a_k \\ T[k-1, s] \vee T[k-1, s-a_k] & \text{if } s \geq a_k \end{cases}$$

for increasing k and then within k increasing s.

Example



Determination of the solution: if $T[k, s] = T[k-1, s]$ then a_k unused and continue with $T[k-1, s]$, otherwise a_k used and continue with $T[k-1, s-a_k]$.

That is mysterious

The algorithm requires a number of $\mathcal{O}(n \cdot z)$ fundamental operations.

What is going on now? Does the algorithm suddenly have polynomial running time?

Explained

The algorithm does not necessarily provide a polynomial run time. z is an *number* and not a *quantity*!

Input length of the algorithm $\cong$ number bits to *reasonably* represent the data. With the number z this would be $\zeta = \log z$.

Consequently the algorithm requires $\mathcal{O}(n \cdot 2^{\zeta})$ fundamental operations and has a run time exponential in ζ.

If, however, z is polynomial in n then the algorithm has polynomial run time in n. This is called *pseudo-polynomial*.

NP

It is known that the subset-sum algorithm belongs to the class of *NP*-complete problems (and is thus *NP-hard*).

P: Set of all problems that can be solved in polynomial time.

NP: Set of all problems that can be solved Nondeterministically in Polynomial time.

Implications:

- NP contains P.
- Problems can be *verified* in polynomial time.
- Under the not (yet?) proven assumption[27] that NP $\neq$ P, there is *no algorithm with polynomial run time* for the problem considered above.

[27] The most important unsolved question of theoretical computer science.

The knapsack problem

We pack our suitcase with ...

- toothbrush
- dumbell set
- coffee machine
- uh oh – too heavy.

- Toothbrush
- Air balloon
- Pocket knife
- identity card
- dumbell set
- Uh oh – too heavy.

- toothbrush
- coffe machine
- pocket knife
- identity card
- Uh oh – too heavy.

Aim to take as much as possible with us. But some things are more valuable than others!

Knapsack problem

Given:

- set of $n \in \mathbb{N}$ items $\{1, \ldots, n\}$.
- Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$.
- Maximum weight $W \in \mathbb{N}$.
- Input is denoted as $E = (v_i, w_i)_{i=1,\ldots,n}$.

Wanted:

a selection $I \subseteq \{1, \ldots, n\}$ that maximises $\sum_{i \in I} v_i$ under $\sum_{i \in I} w_i \leq W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i: Permutation p with $v_{p_i}/w_{p_i} \geq v_{p_{i+1}}/w_{p_{i+1}}$

Add items in this order ($I \leftarrow I \cup \{p_i\}$), if the maximum weight is not exceeded.

That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the selection. But is it good?

Counterexample

$$v_1 = 1 \qquad w_1 = 1 \qquad v_1/w_1 = 1$$

$$v_2 = W - 1 \quad w_2 = W \quad v_2/w_2 = \frac{W-1}{W}$$

Greed algorithm chooses $\{v_1\}$ with value 1.
Best selection: $\{v_2\}$ with value $W - 1$ and weight W.

Greedy heuristics can be arbitrarily bad.

Dynamic Programming

Partition the maximum weight.

Three dimensional table $m[i, w, v]$ ("doable") of boolean values.

$m[i, w, v] =$ true if and only if

- A selection of the first i parts exists $(0 \le i \le n)$
- with overal weight w $(0 \le w \le W)$ and
- a value of at least v $(0 \le v \le \sum_{i=1}^{n} v_i)$.

Computation of the DP table

Initially

- $m[i, w, 0] \leftarrow$ true für alle $i \ge 0$ und alle $w \ge 0$.
- $m[0, w, v] \leftarrow$ false für alle $w \ge 0$ und alle $v > 0$.

Computation

$$m[i, w, v] \leftarrow \begin{cases} m[i-1, w, v] \vee m[i-1, w - w_i, v - v_i] & \text{if } w \ge w_i \text{ und } v \ge v_i \\ m[i-1, w, v] & \text{otherwise.} \end{cases}$$

increasing in i and for each i increasing in w and for fixed i and w increasing by v.

Solution: largest v, such that $m[i, w, v] =$ true for some i and w.

Observation

The definition of the problem obviously implies that

- for $m[i, w, v] =$ true it holds:
  $m[i', w, v] =$ true $\forall i' \ge i$,
  $m[i, w', v] =$ true $\forall w' \ge w$,
  $m[i, w, v'] =$ true $\forall v' \le w$.
- fpr $m[i, w, v] =$ false it holds:
  $m[i', w, v] =$ false $\forall i' \le i$,
  $m[i, w', v] =$ false $\forall w' \le w$,
  $m[i, w, v'] =$ false $\forall v' \ge w$.

This strongly suggests that we do not need a 3d table!

2d DP table

Table entry $t[i,w]$ contains, instead of boolean values, the largest v, that can be achieved[28] with

- items $1,\dots,i$ ($0 \le i \le n$)
- at maximum weight w ($0 \le w \le W$).

[28] We could have followed a similar idea in order to reduce the size of the sparse table.

Computation

Initially

- $t[0,w] \leftarrow 0$ for all $w \ge 0$.

We compute

$$t[i,w] \leftarrow \begin{cases} t[i-1,w] & \text{if } w < w_i \\ \max\{t[i-1,w], t[i-1,w-w_i]+v_i\} & \text{otherwise.} \end{cases}$$

increasing by i and for fixed i increasing by w.

Solution is located in $t[n,w]$

Example

$E = \{(2,3),(4,5),(1,1)\}$

$w \longrightarrow$, $i \downarrow$

	0	1	2	3	4	5	6	7
$\emptyset$	0	0	0	0	0	0	0	0
$(2,3)$	0	0	3	3	3	3	3	3
$(4,5)$	0	0	3	3	5	5	8	8
$(1,1)$	0	1	3	4	5	6	8	9

Reading out the solution: if $t[i,w] = t[i-1,w]$ then item i unused and continue with $t[i-1,w]$ otherwise used and continue with $t[i-1,s-w_i]$.

Analysis

The two algorithms for the knapsack problem provide a run time in $\Theta(n \cdot W \cdot \sum_{i=1}^{n} v_i)$ (3d-table) and $\Theta(n \cdot W)$ (2d-table) and are thus both pseudo-polynomial, but they deliver the best possible result.

The greedy algorithm is very fast butmight deliver an arbitrarily bad result.

Now we consider a solution between the two extremes.

Approximation

Let $\varepsilon \in (0,1)$ given. Let I_{opt} an optimal selection.

No try to find a valid selection I with

$$\sum_{i\in I} v_i \geq (1-\varepsilon)\sum_{i\in I_{\text{opt}}} v_i.$$

Sum of weights may not violate the weight limit.

Different formulation of the algorithm

Before: weight limit $w \to$ maximal value v

Reversal: value $v \to$ minimal weight w

$\Rightarrow$ alternative table $g[i,v]$ provides the minimum weight with

- a selection of the first i items ($0 \leq i \leq n$) that
- provide a value of exactly v ($0 \leq v \leq \sum_{i=1}^{n} v_i$).

Computation

Initially

- $g[0,0] \leftarrow 0$
- $g[0,v] \leftarrow \infty$ (Value v cannot be achieved with 0 items.).

Computation

$$g[i,v] \leftarrow \begin{cases} g[i-1,v] & \text{falls } v < v_i \\ \min\{g[i-1,v], g[i-1,v-v_i]+w_i\} & \text{sonst.} \end{cases}$$

incrementally in i and for fixed i increasing in v.

Solution can be found at largest index v with $g[n,v] \leq w$.

Example

$E = \{(2,3),(4,5),(1,1)\}$

$v \longrightarrow$

$i \downarrow$

	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2,3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4,5)$	0	∞	∞	2	∞	4	∞	∞	6	∞
$(1,1)$	0	1	∞	2	3	4	5	∞	6	7

Read out the solution: if $g[i,v] = g[i-1,v]$ then item i unused and continue with $g[i-1,v]$ otherwise used and continue with $g[i-1,b-v_i]$.

The approximation trick

Pseduopolynomial run time gets polynmial if the number of occuring values can be bounded by a polynom of the input length.

Let $K > 0$ be chosen *appropriately*. Replace values v_i by "rounded values" $\tilde{v}_i = \lfloor v_i/K \rfloor$ delivering a new input $E' = (w_i, \tilde{v}_i)_{i=1\dots n}$.

Apply the algorithm on the input E' with the same weight limit W.

Idea

Example $K = 5$

Values

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots, 98, 99, 100$$
$$\rightarrow$$
$$0, 0, 0, 0, 1, 1, 1, 1, 1, 2, \dots, 19, 19, 20$$

Obviously less different values

Properties of the new algorithm

- Selection of items in E' is also admissible in E. Weight remains unchanged!
- Run time of the algorithm is bounded by $\mathcal{O}(n^2 \cdot v_{\max}/K)$ ($v_{\max} := \max\{v_i | 1 \leq i \leq n\}$)

How good is the approximation?

It holds that

$$v_i - K \leq K \cdot \left\lfloor \frac{v_i}{K} \right\rfloor = K \cdot \tilde{v}_i \leq v_i$$

Let I'_{opt} be an optimal solution of E'. Then

$$\left(\sum_{i \in I_{\text{opt}}} v_i \right) - n \cdot K \overset{|I_{\text{opt}}| \leq n}{\leq} \sum_{i \in I_{\text{opt}}} (v_i - K) \leq \sum_{i \in I_{\text{opt}}} (K \cdot \tilde{v}_i) = K \sum_{i \in I_{\text{opt}}} \tilde{v}_i$$
$$\underset{I'_{\text{opt}}\text{optimal}}{\leq} K \sum_{i \in I'_{\text{opt}}} \tilde{v}_i = \sum_{i \in I'_{\text{opt}}} K \cdot \tilde{v}_i \leq \sum_{i \in I'_{\text{opt}}} v_i.$$

Choice of K

Requirement:

$$\sum_{i\in I'} v_i \geq (1-\varepsilon)\sum_{i\in I_{\text{opt}}} v_i.$$

Inequality from above:

$$\sum_{i\in I'_{\text{opt}}} v_i \geq \left(\sum_{i\in I_{\text{opt}}} v_i\right) - n\cdot K$$

thus: $K = \varepsilon\frac{\sum_{i\in I_{\text{opt}}} v_i}{n}$.

Choice of K

Choose $K = \varepsilon\frac{\sum_{i\in I_{\text{opt}}} v_i}{n}$. The optimal sum is unknown. Therefore we choose $K' = \varepsilon\frac{v_{\max}}{n}$.[29]

It holds that $v_{\max} \leq \sum_{i\in I_{\text{opt}}} v_i$ and thus $K' \leq K$ and the approximation is even slightly better.

The run time of the algorithm is bounded by

$$\mathcal{O}(n^2\cdot v_{\max}/K') = \mathcal{O}(n^2\cdot v_{\max}/(\varepsilon\cdot v_{\max}/n)) = \mathcal{O}(n^3/\varepsilon).$$

[29] We can assume that items i with $w_i > W$ have been removed in the first place.

FPTAS

Such a family of algorithms is called an *approximation scheme*: the choice of ε controls both running time and approximation quality.

The runtime $\mathcal{O}(n^3/\varepsilon)$ is a polynom in n and in $\frac{1}{\varepsilon}$. The scheme is therefore also called a *FPTAS - Fully Polynomial Time Approximation Scheme*

Optimal binary Search Trees

Given: search probabilities p_i for each key k_i ($i = 1,\ldots,n$) and q_i of each interval d_i ($i = 0,\ldots,n$) between search keys of a binary search tree. $\sum_{i=1}^n p_i + \sum_{i=0}^n q_i = 1$.

Wanted: optimal search tree T with key depths $\text{depth}(\cdot)$, that minimizes the expected search costs

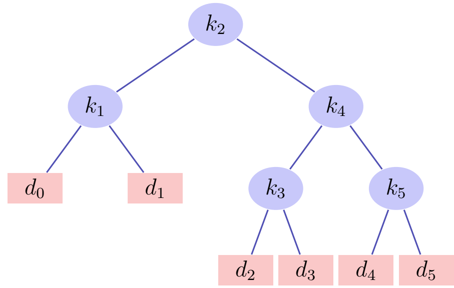
$$\begin{aligned}C(T) &= \sum_{i=1}^n p_i\cdot(\text{depth}(k_i)+1) + \sum_{i=0}^n q_i\cdot(\text{depth}(d_i)+1)\\ &= 1 + \sum_{i=1}^n p_i\cdot\text{depth}(k_i) + \sum_{i=0}^n q_i\cdot\text{depth}(d_i)\end{aligned}$$

Example

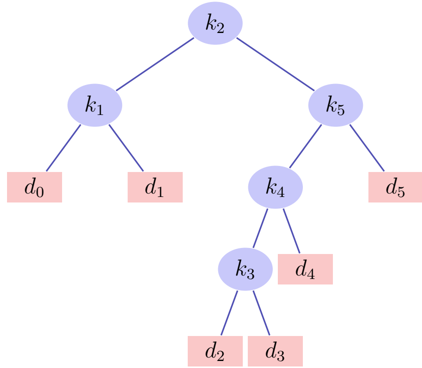
Expected Frequencies

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Example



Search tree with expected costs 2.8



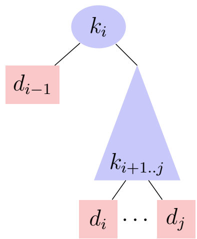
Search tree with expected costs 2.75

Structure of a optimal binary search tree

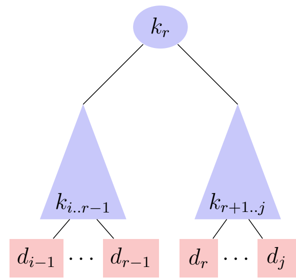
- Subtree with keys $k_i, \dots, k_j$ and intervals $d_{i-1}, \dots, d_j$ must be optimal for the respective sub-problem.[30]
- Consider all subtrees with roots k_r and optimal subtrees for keys $k_i, \dots, k_{r-1}$ and $k_{r+1}, \dots, k_j$

[30] The usual argument: if it was not optimal, it could be replaced by a better solution improving the overal solution.

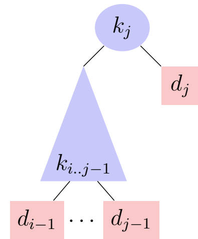
Sub-trees for Searching



empty left subtree



non-empty left and right subtrees



empty right subtree

Expected Search Costs

Let $\text{depth}_T(k)$ be the depth of a node k in the sub-tree T. Let k be the root of subtrees T_r and T_{L_r} and T_{R_r} be the left and right sub-tree of T_r. Then

$$\begin{aligned}\text{depth}_T(k_i) &= \text{depth}_{T_{L_r}}(k_i) + 1,\ (i < r)\\ \text{depth}_T(k_i) &= \text{depth}_{T_{R_r}}(k_i) + 1,\ (i > r)\end{aligned}$$

Expected Search Costs

Let $e[i, j]$ be the costs of an optimal search tree with nodes $k_i, \ldots, k_j$.

Base case $e[i, i-1]$, expected costs d_{i-1}

Let $w(i, j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$.

If k_r is the root of an optimal search tree with keys $k_i, \ldots, k_j$, then

$$e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$$

with $w(i, j) = w(i, r-1) + p_r + w(r+1, j)$:

$$e[i, j] = e[i, r-1] + e[r+1, j] + w(i, j).$$

Dynamic Programming

$$e[i, j] = \begin{cases} q_{i-1} & \text{if } j = i-1,\\ \min_{i \le r \le j}\{e[i, r-1] + e[r+1, j] + w[i, j]\} & \text{if } i \le j \end{cases}$$

Computation

Tables $e[1 \ldots n+1, 0 \ldots n], w[1 \ldots n+1, 0 \ldots m], r[1 \ldots n, 1 \ldots n]$
Initially

- $e[i, i-1] \leftarrow q_{i-1}$, $w[i, i-1] \leftarrow q_{i-1}$ for all $1 \le i \le n+1$.

We compute

$$\begin{aligned} w[i, j] &= w[i, j-1] + p_j + q_j\\ e[i, j] &= \min_{i \le r \le j}\{e[i, r-1] + e[r+1, j] + w[i, j]\}\\ r[i, j] &= \arg\min_{i \le r \le j}\{e[i, r-1] + e[r+1, j] + w[i, j]\} \end{aligned}$$

for intervals $[i, j]$ with increasing lengths $l = 1, \ldots, n$, each for $i = 1, \ldots, n-l+1$. Result in $e[1, n]$, reconstruction via r. Runtime $\Theta(n^3)$.

Example

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

w

j \ i	1	2	3	4	5	6
0	0.05					
1	0.30	0.10				
2	0.45	0.25	0.05			
3	0.55	0.35	0.15	0.05		
4	0.70	0.50	0.30	0.20	0.05	
5	1.00	0.80	0.60	0.50	0.35	0.10

e

j \ i	1	2	3	4	5	6
0	0.05					
1	0.45	0.10				
2	0.90	0.40	0.05			
3	1.25	0.70	0.25	0.05		
4	1.75	1.20	0.60	0.30	0.05	
5	2.75	2.00	1.30	0.90	0.50	0.10

r

j \ i	1	2	3	4	5
1	1				
2	1	2			
3	2	2	3		
4	2	2	4	4	
5	2	4	5	5	5