19. Dynamic Programming I

Fibonacci, Längste aufsteigende Teilfolge, längste gemeinsame Teilfolge, Editierdistanz, Matrixkettenmultiplikation, Matrixmultiplikation nach Strassen [Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

Fibonacci Numbers

(again)

$$F_n := \begin{cases} 1 & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 3. \end{cases}$$

Analysis: why ist the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n \leq 2$ **then**
  $f \leftarrow 1$
else
  $f \leftarrow$ FibonacciRecursive$(n-1)$ + FibonacciRecursive$(n-2)$
return f

Analysis

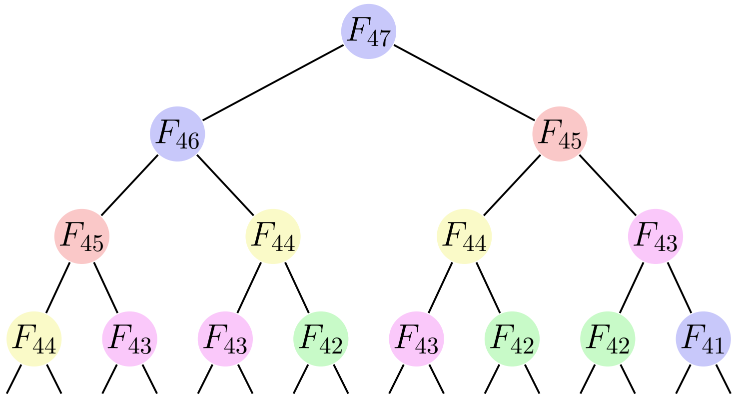
$T(n)$: Number executed operations.

- $n = 1, 2$: $T(n) = \Theta(1)$
- $n \geq 3$: $T(n) = T(n-2) + T(n-1) + c$.

$$T(n) = T(n-2) + T(n-1) + c \geq 2T(n-2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

Algorithm is *exponential* in n.

Reason (visual)



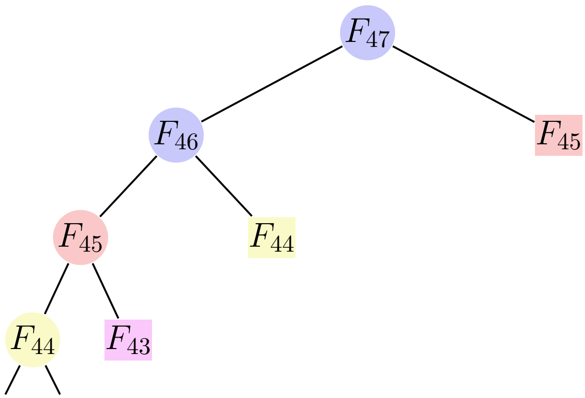
Nodes with same values are evaluated often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rechteckige Knoten wurden bereits ausgewertet.

Algorithm FibonacciMemoization(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n \leq 2$ **then**
    $f \leftarrow 1$
else if $\exists$memo$[n]$ **then**
    $f \leftarrow$ memo$[n]$
else
    $f \leftarrow$ FibonacciMemoization$(n-1)$ + FibonacciMemoization$(n-2)$
    memo$[n] \leftarrow f$
return f

Analysis

Computational complexity:

$$T(n) = T(n-1) + c = ... = \mathcal{O}(n).$$

Algorithm requires $\Theta(n)$ memory.[24]

[24] But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3,... in the *top-down* approach of the recursion.

Can write the algorithm *bottom-up*. Then it is called *dynamic programming*.

Algorithm FibonacciDynamicProgram(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

$F[1] \leftarrow 1$
$F[2] \leftarrow 1$
for $i \leftarrow 3, \ldots, n$ **do**
  $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

Dynamic Programming: Procedure

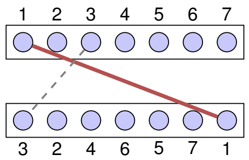
1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?
3. Determine *computation order*.
   In which order can the entries be computed such that dependencies are fulfilled?
4. Read-out the *solution*
   How can the solution be read out from the table?

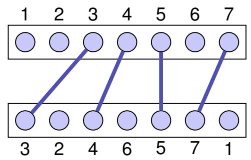
Runtime (typical) = number entries of the table times required operations per entry.

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**
   $n \times 1$ table. nth entry contains nth Fibonacci number.
2. **Which entries do not depend on other entries?**
   Values F_1 and F_2 can be computed easily and independently.
3. **What is the execution order such that required entries are always available?**
   F_i with increasing i.
4. **Wie kann sich Lösung aus der Tabelle konstruieren lassen?**
   F_n ist die n-te Fibonacci-Zahl.

Longest Ascending Sequence (LAS)

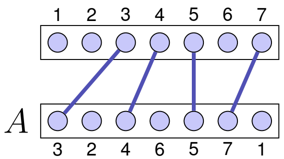




Connect as many as possible fitting ports without lines crossing.

Formally

- Consider Sequence $A = (a_1, \dots, a_n)$.
- Search for a longest increasing subsequence of A.
- Examples of increasing subsequences: $(3,4,5)$, $(2,4,5,7)$, $(3,4,5,7)$, $(3,7)$.



Generalization: allow any numbers, even with duplicates. But only strictly increasing subsequences are permitted. Example: $(2,3,3,3,5,1)$ with increasing subsequence $(2,3,5)$.

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k+1$.

If a_{k+1} fits to L_k, then $L_{k+1} = L_k \oplus a_{k+1}$

Counterexample $A_5 = (1,2,5,3,4)$. Let $A_3 = (1,2,5)$ with $L_3 = A$. Determine L_4 from L_3?

It does not work this way, we cannot infer L_{k+1} from L_k.

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

Counterexample: $A_5 = (1,2,5,3,4)$. Let $A_4 = (1,2,5,3)$ with $L_1 = (1)$, $L_2 = (1,2)$, $L_3 = (1,2,5)$, $L_4 = (1,2,5)$. Determine L_5 from $L_1, \dots, L_4$?

That does not work either: cannot infer L_{k+1} from only *an arbitrary solution* L_j. We need to consider all LAS. Too many.

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$
$(1, 1000, 1001)$	$(1), (1, 1000), (1, 1000, 1001)$
$(1, 1000, 1001, 2)$	$(1), (1, 2), (1, 1000, 1001)$
$(1, 1000, 1001, 2, 3)$	$(1), (1, 2), (1, 2, 3)$

DP Table

- Idea: save the last element of an increasing sequence at slot j.
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4
Predecessor	$-\infty$	$-\infty$	2	$-\infty$	5	1

0	1	2	3	4	...
-∞	∞	∞	∞	∞	
-∞	3	∞	∞	∞	
-∞	2	∞	∞	∞	
-∞	2	5	∞	∞	
-∞	1	5	∞	∞	
-∞	1	5	6	∞	
-∞	1	4	6	∞	

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1. Two tables $T[0, \dots, n]$ and $V[1, \dots, n]$. Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Computation of an entry

2. Entries in T sorted in ascending order. For each new entry a_{k+1} binary search for l, such that $T[l] < a_k < T[l+1]$. Set $T[l+1] \leftarrow a_{k+1}$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3 **Computation order**

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

4 **How can the solution be determined from the table?**

Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = A[i]$, i is the predecessor of l. Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

- Computation of the table:
  - Initialization: $\Theta(n)$ Operations
  - Computation of the kth entry: binary search on positions $\{1, \ldots, k\}$ plus constant number of assignments.

$$\sum_{k=1}^{n} (\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^{n} \log(k) = \Theta(n \log n).$$

- Reconstruction: traverse A from right to left: $\mathcal{O}(n)$.

Overal runtime:

$$\Theta(n \log n).$$

Longest common subsequence

Subsequences of a string:

Subsequences(KUH): (), (K), (U), (H), (KU), (KH), (UH), (KUH)

Problem:

- Input: two strings $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_n)$ with lengths $m > 0$ and $n > 0$.
- Wanted: Longest common subsequecnes (LCS) of A and B.

Application: DNA sequence alignment.

Longest Common Subsequence

Examples:

LGT(IGEL,KATZE)=E, LGT(TIGER,ZIEGE)=IGE

Ideas to solve?

T	$\boxed{\text{I}}$		$\boxed{\text{G}}$	$\boxed{\text{E}}$	R
Z	$\boxed{\text{I}}$	E	$\boxed{\text{G}}$	$\boxed{\text{E}}$	

Recursive Procedure

Assumption: solutions $L(i, j)$ known for $A[1, \ldots, i]$ and $B[1, \ldots, j]$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, but not for $i = m$ and $j = n$.

T	$\boxed{\text{I}}$		$\boxed{\text{G}}$	$\boxed{\text{E}}$	R
Z	$\boxed{\text{I}}$	E	$\boxed{\text{G}}$	$\boxed{\text{E}}$	

Consider characters a_m, b_n. Three possibilities:

1. A is enlarged by one whitespace. $L(m, n) = L(m, n-1)$
2. B is enlarged by one whitespace. $L(m, n) = L(m-1, n)$
3. $L(m, n) = L(m-1, n-1) + \delta_{mn}$ with $\delta_{mn} = 1$ if $a_m = b_n$ and $\delta_{mn} = 0$ otherwise

Recursion

$$L(m, n) \leftarrow \max\{L(m-1, n-1) + \delta_{mn}, L(m, n-1), L(m-1, n)\}$$

for $m, n > 0$ and base cases $L(\cdot, 0) = 0$, $L(0, \cdot) = 0$.

	$\emptyset$	Z	I	E	G	E
$\emptyset$	0	0	0	0	0	0
T	0	0	0	0	0	0
I	0	0	1	1	1	1
G	0	0	1	1	2	2
E	0	0	1	2	2	3
R	0	0	1	2	2	3

Dynamic Programming algorithm LCS

1. **Dimension of the table? Semantics?**
   Table $L[0, \ldots, m][0, \ldots, n]$. $L[i, j]$: length of a LCS of the strings $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_j)$
2. **Computation of an entry**
   $L[0, i] \leftarrow 0 \; \forall 0 \leq i \leq m$, $L[j, 0] \leftarrow 0 \; \forall 0 \leq j \leq n$. Computation of $L[i, j]$ otherwise via $L[i, j] = \max(L[i-1, j-1] + \delta_{ij}, L[i, j-1], L[i-1, j])$.

Dynamic Programming algorithm LCS

3. **Computation order**
   Rows increasing and within columns increasing (or the other way round).
4. **Reconstruct solution?**
   Start with $j = m$, $i = n$. If $a_i = b_j$ then output a_i otherwise, if $L[i, j] = L[i, j-1]$ continue with $j \leftarrow j - 1$ otherwise if $L[i, j] = L[i-1, j]$ continue with $i \leftarrow i - 1$. Terminate for $i = 0$ or $j = 0$.

Analysis LCS

- Number table entries: $(m+1)\cdot(n+1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solition: decrease i or j. Maximally $\mathcal{O}(n+m)$ steps.

Runtime overal:

$$\mathcal{O}(mn).$$

Editing Distance

Editing distance of two sequences $A=(a_1,\dots,a_m)$, $B=(b_1,\dots,b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

TIGER ZIGER ZIEGER ZIEGE

Editing Distance = Levenshtein Distance

Procedure?

- Two dimensional table $E[0,\dots,m][0,\dots,n]$ with editing distances $E[i,j]$ of strings $A_i=(a_1,\dots,a_i)$ and $B_j=(b_1,\dots,b_j)$.
- Consider the last characters of A_i and B_j. Three possible cases:
  1. Delete last character of A_i: [25] $E[i-1,j]+1$.
  2. Append character to A_i:[26] $E[i,j-1]+1$.
  3. Replace A_i by B_j: $E[i-1,j-1]+1-\delta_{ij}$.

$$E[i,j]\leftarrow\min\left\{E[i-1,j]+1,E[i,j-1]+1,E[i-1,j-1]+1-\delta_{ij}\right\}$$

[25] or append character to B_j

[26] or delete last character of B_j

DP Table

$$E[i,j]\leftarrow\min\left\{E[i-1,j]+1,E[i,j-1]+1,E[i-1,j-1]+1-\delta_{ij}\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Algorithm: exercise

Matrix-Chain-Multiplication

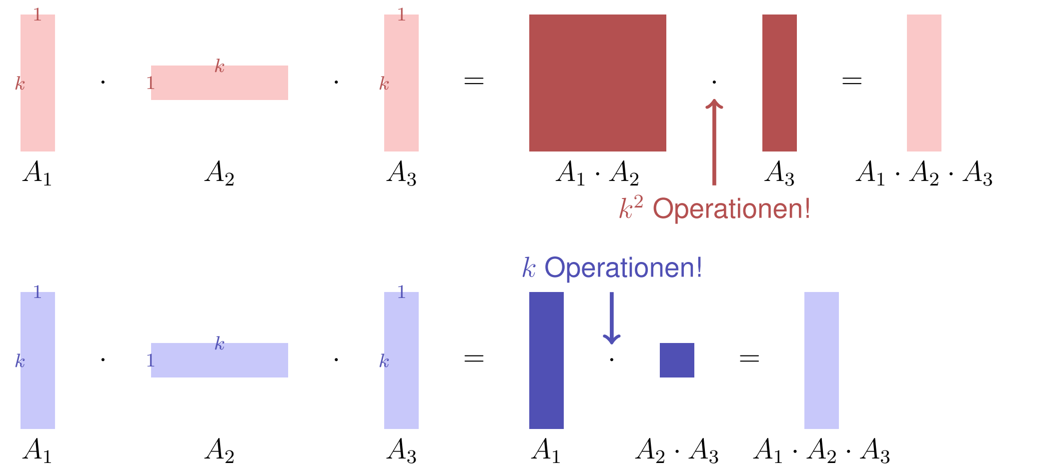
Task: Computation of the product $A_1 \cdot A_2 \cdot ... \cdot A_n$ of matrices $A_1, \dots, A_n$.

Matrix multiplication is associative, d.h. the order of execution can be chosen arbitrarily

Goal: efficient computation of the product.

Assumption: multiplicaiton of an $(r \times s)$-matrix with an $(s \times u)$-matrix provides costs $r \cdot s \cdot u$.

Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i.
- Compute best i, done.

$n \times n$-table M. entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \le i < p} \left(M[p, i] + M[i + 1, q] + \text{costs of the last multiplication} \right)$$

Computation of the DP-table

- Base cases $M[p, p] \leftarrow 0$ for all $1 \le p \le n$.
- Computation of $M[p, q]$ depends on $M[i, j]$ with $p \le i \le j \le q$, $(i, j) \ne (p, q)$.
  In particular $M[p, q]$ depends at most from entries $M[i, j]$ with $i - j < q - p$.
  Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n-1$ other entries.

Overal runtime $\mathcal{O}(n^3)$.

Readout the order from M: exercise!

Digression: matrix multiplication

Consider the mutliplicaiton of two $n \times n$ matrices.

Let

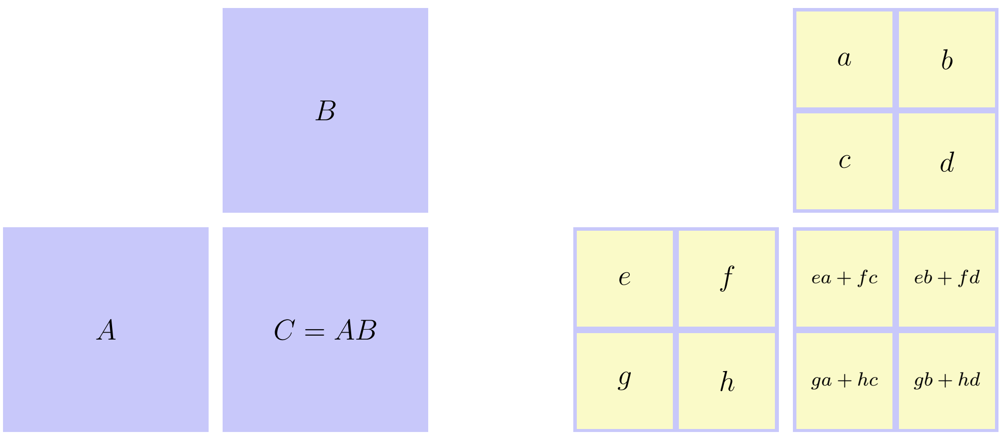
$$A = (a_{ij})_{1\le i,j\le n}, B = (b_{ij})_{1\le i,j\le n}, C = (c_{ij})_{1\le i,j\le n},$$
$$C = A \cdot B$$

then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

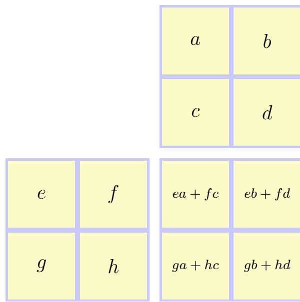
Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

Divide and Conquer



Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications: $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage ☹



Strassen's Matrix Multiplication

- Nontrivial observation by Strassen (1969):
  It suffices to compute the seven products
  $A = (e + h) \cdot (a + d)$, $B = (g + h) \cdot a$,
  $C = e \cdot (b - d)$, $D = h \cdot (c - a)$, $E = (e + f) \cdot d$,
  $F = (g - e) \cdot (a + b)$, $G = (f - h) \cdot (c + d)$. Denn:
  $ea + fc = A + D - E + G$, $eb + fd = C + E$,
  $ga + hc = B + D$, $gb + hd = A - B + C + F$.
- This yields $M'(n) = 7M(n/2)$, $M'(1) = 1$.
  Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.
- Fastest currently known algorithm:
  $\mathcal{O}(n^{2.37})$