Datenstrukturen und Algorithmen

Vorlesung am D-Math (CSE) der ETH Zürich

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FS 2017

Willkommen!

Course homepage

`http://lec.inf.ethz.ch/DA/2017`

The team:

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Dozent	Felix Friedrich

1. Introduction

Algorithms and Data Structures, Three Examples

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.

Goals of the course

On the one hand

- Essential basic knowlegde from computer science.

Andererseits

- Preparation for your further course of studies and practical considerations.

Contents

data structures / algorithms

The notion invariant, cost model, Landau notation

algorithms design, induction

searching, selection and sorting

dynamic programming

dictionaries: hashing and search trees

graphs, shortest paths, backtracking, flow

geometric algorithms, high peformance LA

Randomized algorithms (Gibbs/SA), multiscale approach

sorting networks, parallel algorithms

prorgamming with C++

RAII, Move Konstruktion, Smart Pointers,

Templates and generic programming

Exceptions

Constexpr, user defined literals

functors and lambdas

threads, mutex and monitors

promises and futures

parallel programming

parallelism vs. concurrency, speedup (Amdahl/-Gustavson), races, memory reordering, atomir registers, RMW (CAS,TAS), deadlock/starvation

literature

Algorithmen und Datenstrukturen, *T. Ottmann, P. Widmayer*, Spektrum-Verlag, 5. Auflage, 2011

Algorithmen - Eine Einführung, *T. Cormen, C. Leiserson, R. Rivest, C. Stein*, Oldenbourg, 2010

Introduction to Algorithms, *T. Cormen, C. Leiserson, R. Rivest, C. Stein* , 3rd ed., MIT Press, 2009

The C++ Programming Language, *B. Stroustrup*, 4th ed., Addison-Wesley, 2013.

The Art of Multiprocessor Programming, *M. Herlihy, N. Shavit*, Elsevier, 2012.

1.2 Algorithms

[Cormen et al, Kap. 1;Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm: well defined computing procedure to compute *output* data from *input* data

example problem

Input : A sequence of n numbers $(a_1, a_2, \ldots, a_n)$

Output : Permutation $(a'_1, a'_2, \ldots, a'_n)$ of the sequence $(a_i)_{1 \leq i \leq n}$, such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) ...

Every example represents a *problem instance*

Examples for algorithmic problems

- routing: shortest path
- cryptography / digital signatures
- time table / working plans: linear programming
- DNA matching: dynamic programming
- fabrication pipeline: topological sort
- geometric probelms, e.g. convex hull

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Darta Structures

- Organisation of the data tailored towards the algorithms that operate on the data.
- Programs = algorithms + data structures.

Very hard problems.

- NP-compleete problems: no known efficient solution (but the non-existence of such a solution is not proven yet!)
- Example: travelling salesman problem

A dream

- If computers were infinitely fast and had an infinite amount of memory ...
- ... then we would still need the theory of algorithms (only) for statements about correctness (and termination).

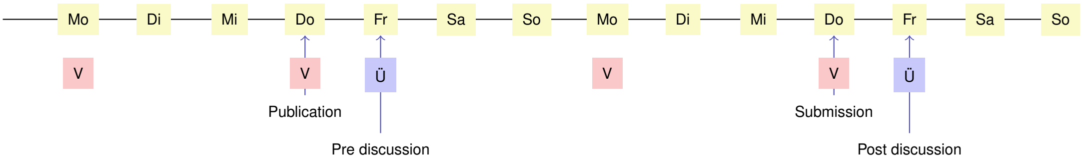
The reality

Resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

1.3 Organisation

The exercise process



- Exercise publication each Thursday
- Preliminary discussion on Friday
- Latest submission Thursday one week later
- Debriefing of the exercise on follong Friday. Feedback to your submissions within a week after debriefing.

Codeboard

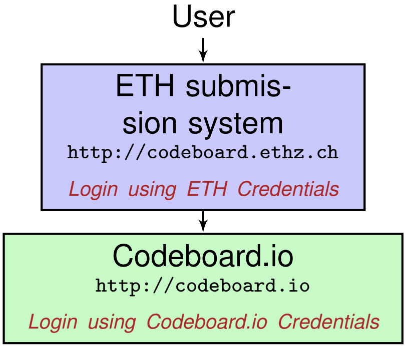
Codeboard is an online-IDE: programming in the browser

- Examples can be tried without any tool installation.
- Used for the exercises.

Codeboard @ETH

Codeboard consists of two independent communicating systems:

- **The ETH submission system** Allows us to correct you submissions
- **The online IDE** The programming environment.



Codeboard

Codeboard.io registration

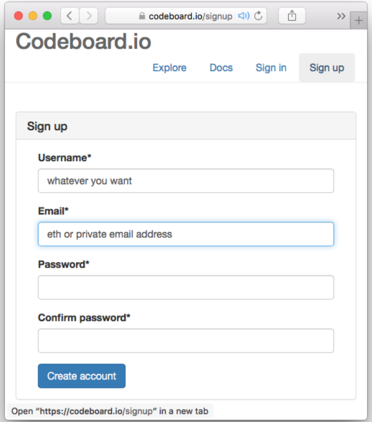
Go to `http://codeboard.io` and create an account, best is to stay logged in

Register for the recitation sessions

Go to `http://codeboard.ethz.ch/da` and register for a recitation session there.

Codeboard.io registration

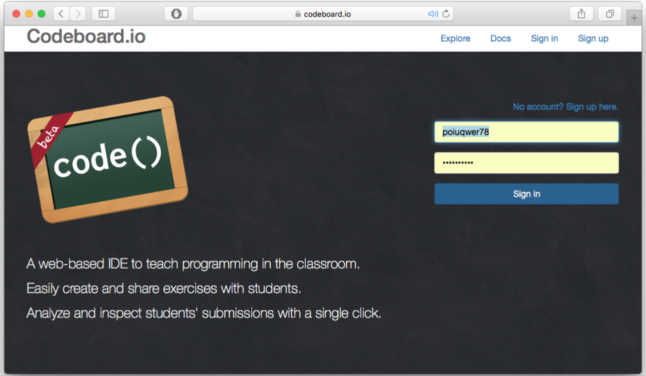
Should you not yet have a **Codeboard.io** account ...



- We will be using the online IDE **Codeboard.io**
- create an account in order to be able to store your progress
- Login data can be chose arbitrarily. *Do not use your ETH password.*

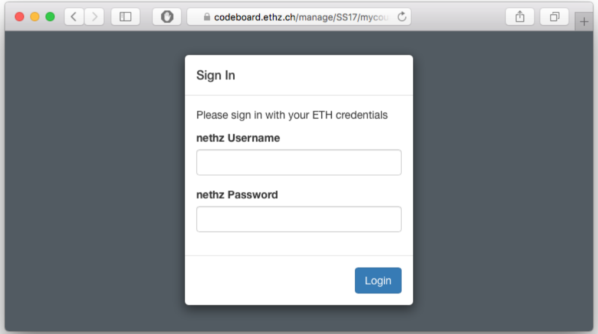
Codeboard.io Login

If you have an account, log in:



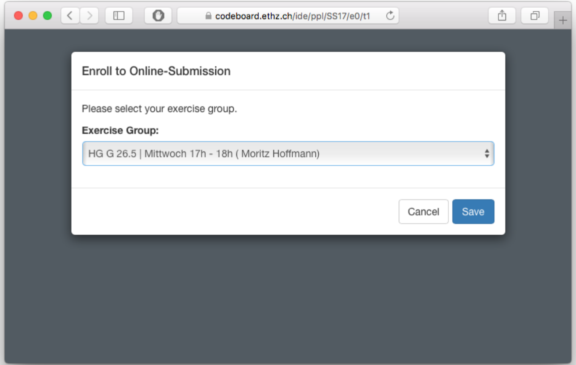
Recitation session registration - I

- Visit `http://codeboard.ethz.ch/da`
- Login with your ETH account



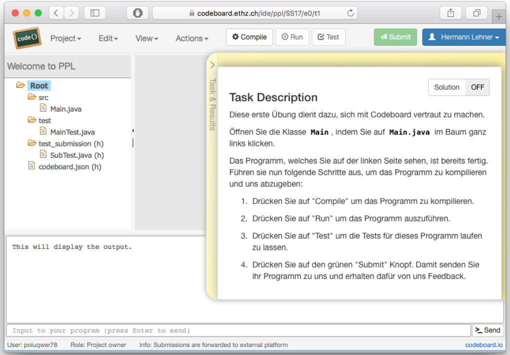
Recitation session registration - II

Register using the dialog with a recitation session.



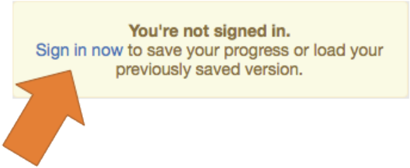
The first exercise

You are now registered and the first exercise is loaded. Follow the guidelines in the yellow box. The exercise sheet on the course homepage contains further instructions and explanations.



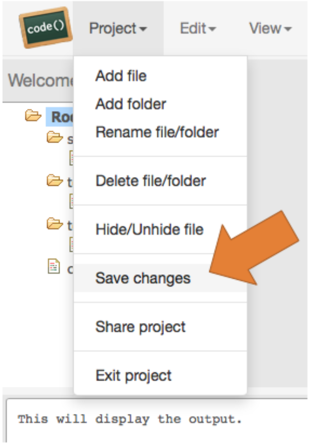
The first exercise – Codeboard.io Login

If you see this message, click on Sign in now and log in with your **Codeboard.io** account.



The first exercise – store progress!

Attention! Store your progress on a regular basis. The you can continue somewhere else easily.



About the exercises

- Since HS 2013 no exercise certificate required any more for exam admission
- Doing the exercises and going to the recitation sessions is optional but highly recommended!

Relevant for the exam

Material for the exam comprises

- Course content (lectures, handout)
- Exercises content (exercise sheets, recitation hours)

Written exam (120 min). Examination aids: four A4 pages (or two sheets of 2 A4 pages double sided) either hand written or with font size minimally 11 pt.

In your and our interest

Please let us know early if you see any problems, if

- the lectures are too fast, too difficult, too ...
- the exercises are not doable or not understandable ...
- you do not feel well supported ...

In short: if you have
any issues that we can fix.

1.4 Ancient Egyptian Multiplication

Ancient Egyptian Multiplication

Example 1: Ancient Egyptian Multiplication[1]

Compute $11 \cdot 9$

$$
\begin{array}{r|r}
11 & 9 \\
\cancel{22} & \cancel{4} \\
\cancel{44} & \cancel{2} \\
88 & 1 \\
\hline
99 & -
\end{array}
\qquad
\begin{array}{r|r}
9 & 11 \\
18 & 5 \\
\cancel{36} & \cancel{2} \\
72 & 1 \\
\hline
99 &
\end{array}
$$

1. Double left, integer division by 2 on the right
2. Even number on the right $\Rightarrow$ eliminate row.
3. Add remaining rows on the left.

[1] Also known as russian multiplication

Advantages

- Short description, easy to grasp
- Efficient to implement on a computer: double = left shift, divide by 2 = right shift

Beispiel

left shift	$9 = 01001_2 \to 10010_2 = 18$
right shift	$9 = 01001_2 \to 00100_2 = 4$

Questions

- Does this always work (negative numbers?)?
- If not, when does it work?
- How do you prove correctness?
- Is it better than the school method?
- What does “good” mean at all?
- How to write this down precisely?

Observation

If $b > 1$, $a \in \mathbb{Z}$, then:

$$a \cdot b = \begin{cases} 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade,} \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade.} \end{cases}$$

Termination

$$a \cdot b = \begin{cases} a & \text{falls } b = 1, \\ 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade}, \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade}. \end{cases}$$

Recursively, Functional

$$f(a,b) = \begin{cases} a & \text{falls } b = 1, \\ f(2a, \frac{b}{2}) & \text{falls } b \text{ gerade,} \\ a + f(2a, \frac{b-1}{2}) & \text{falls } b \text{ ungerade.} \end{cases}$$

Implemented

```
// pre: b>0
// post: return a*b
int f(int a, int b){
    if(b==1)
        return a;
    else if (b%2 == 0)
        return f(2*a, b/2);
    else
        return a + f(2*a, (b-1)/2);
}
```

Correctnes

$$f(a,b) = \begin{cases} a & \text{if } b = 1, \\ f(2a, \frac{b}{2}) & \text{if } b \text{ even,} \\ a + f(2a \cdot \frac{b-1}{2}) & \text{if } b \text{ odd.} \end{cases}$$

Remaining to show: $f(a,b) = a \cdot b$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}^+$.

Proof by induction

Base clause: $b = 1 \Rightarrow f(a, b) = a = a \cdot 1$.
Hypothesis: $f(a, b') = a \cdot b'$ für $0 < b' \leq b$
Step: $f(a, b+1) \overset{!}{=} a \cdot (b+1)$

$$f(a, b+1) = \begin{cases} f(2a, \overbrace{\frac{b+1}{2}}^{\leq b}) = a \cdot (b+1) & \text{if } b \text{ odd,} \\ a + f(2a, \underbrace{\frac{b}{2}}_{\leq b}) = a + a \cdot b & \text{if } b \text{ even.} \end{cases}$$

■

End Recursion

The recursion can be writen as *end recursion*

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  else if (b%2 == 0)
    return f(2*a, b/2);
  else
    return a + f(2*a, (b-1)/2);
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```

End-Recursion ⇒ Iteration

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```

→

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b;
      z = a;
    }
    res += z;
    a *= 2; // neues a
    b /= 2; // neues b
  }
  res += a; // Basisfall b=1
  return res;
}
```

Simplify

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b;   ⟶ Teil der Division
      z = a; ⟶ Direkt in res
    }
    res += z;
    a *= 2;
    b /= 2;
  }
  res += a;  ⟶ in den Loop
  return res;
}
```

⟶

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0)
      res += a;
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Invariants!

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0){
      res += a;
      --b;
    }
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Sei $x = a \cdot b$.

here: $x = \boxed{a \cdot b + res}$

if here $x = a \cdot b + res$ ...

... then also here $x = a \cdot b + res$

b even

here: $x = a \cdot b + res$

here: $x = a \cdot b + res$ und $b = 0$

Also $res = x$.

Conclusion

The expression $a \cdot b + res$ is an *invariant*

- Values of a, b, res change but the invariant remains basically unchanged
- The invariant is only temporarily discarded by some statement but then re-established
- If such short statement sequences are considered atomiv, the value remains indeed invariant
- In particular the loop contains an invariant, called *loop invariant* and operates there like the induction step in induction proofs.
- Invariants are obviously powerful tools for proofs!

Further simplification

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0)
      res += a;
    a *= 2;
    b /= 2;
  }
  return res;
}
```

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Analysis

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Ancient Egyptian Multiplication corresponds to the school method with radix 2.

$$
\begin{array}{ccccccccc}
 & 1 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 & \\
\hline
 & & & & & & 1 & 0 & 0 & 1 & (9) \\
 & & & & & 1 & 0 & 0 & 1 & & (18) \\
\hline
 & & & & & 1 & 1 & 0 & 1 & 1 & \\
 & & 1 & 0 & 0 & 1 & & & & & (72) \\
\hline
 & & 1 & 1 & 0 & 0 & 0 & 1 & 1 & & (99)
\end{array}
$$

Efficiency

Question: how long does a multiplication of a and b take?

- Measure for efficiency
  - Total number of fundamental operations: double, divide by 2, shift, test for "even", addition
  - In the recursive code: maximally 6 operations per call
- Essential criterion:
  - Number of recursion calls or
  - Number iterations (in the iterative case)
- $\frac{b}{2^n} \leq 1$ holds for $n \geq \log_2 b$. Consequently not more than $6\lceil \log_2 b \rceil$ fundamental operations.

1.5 Fast Integer Multiplication

[Ottman/Widmayer, Kap. 1.2.3]

Example 2: Multiplication of large Numbers

Primary school:

$$
\begin{array}{cccccc|c}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d \cdot b \\
 & & & 4 & 2 & & d \cdot a \\
 & & & & 6 & & c \cdot b \\
 & & 1 & 8 & & & c \cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$2 \cdot 2 = 4$ single-digit multiplications. $\Rightarrow$ Multiplication of two n-digit numbers: n^2 single-digit multiplications

Observation

$$
\begin{aligned}
ab \cdot cd &= (10 \cdot a + b) \cdot (10 \cdot c + d) \\
&= 100 \cdot a \cdot c + 10 \cdot a \cdot c \\
&\quad + 10 \cdot b \cdot d + b \cdot d \\
&\quad + 10 \cdot (a - b) \cdot (d - c)
\end{aligned}
$$

Improvement?

$$
\begin{array}{r cccc|l}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d \cdot b \\
 & & & 1 & 4 & & d \cdot b \\
 & & & 1 & 6 & & (a-b)\cdot(d-c) \\
 & & & 1 & 8 & & c \cdot a \\
 & & 1 & 8 & & & c \cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$\rightarrow 3$ single-digit multiplications.

Large Numbers

$$6237 \cdot 5898 = \underbrace{62}_{a'}\underbrace{37}_{b'} \cdot \underbrace{58}_{c'}\underbrace{98}_{d'}$$

Recursive / inductive application: compute $a' \cdot c'$, $a' \cdot d'$, $b' \cdot c'$ and $c' \cdot d'$ as shown above.

$\rightarrow 3 \cdot 3 = 9$ instead of 16 single-digit multiplications.

Generalization

Assumption: two numbers with n digits each, $n = 2^k$ for some k.

$$
\begin{aligned}
(10^{n/2}a + b) \cdot (10^{n/2}c + d) &= 10^n \cdot a \cdot c + 10^{n/2} \cdot a \cdot c \\
&\quad + 10^{n/2} \cdot b \cdot d + b \cdot d \\
&\quad + 10^{n/2} \cdot (a - b) \cdot (d - c)
\end{aligned}
$$

Recursive application of this formula: algorithm by Karatsuba and Ofman (1962).

Analysis

$M(n)$: Number of single-digit multiplications.

Recursive application of the algorithm from above $\Rightarrow$ recursion equality:

$$M(2^k) = \begin{cases} 1 & \text{if } k = 0, \\ 3 \cdot M(2^{k-1}) & \text{if } k > 0. \end{cases}$$

Iterative Substition

Iterative substition of the recursion formula in order to guess a solution of the recursion formula:

$$
\begin{aligned}
M(2^k) &= 3 \cdot M(2^{k-1}) = 3 \cdot 3 \cdot M(2^{k-2}) = 3^2 \cdot M(2^{k-2}) \\
&= \dots \\
&\overset{!}{=} 3^k \cdot M(2^0) = 3^k.
\end{aligned}
$$

Proof: induction

Hypothesis H:

$$M(2^k) = 3^k.$$

Base clause ($k = 0$):

$$M(2^0) = 3^0 = 1. \quad \checkmark$$

Induction step ($k \to k+1$):

$$M(2^{k+1}) \stackrel{\text{def}}{=} 3 \cdot M(2^k) \stackrel{\text{H}}{=} 3 \cdot 3^k = 3^{k+1}.$$

$\blacksquare$

Comparison

Traditionally n^2 single-digit multiplications.

Karatsuba/Ofman:

$$M(n) = 3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = 2^{\log_2 3 \log_2 n} = n^{\log_2 3} \approx n^{1.58}.$$

Example: number with 1000 digits: $1000^2/1000^{1.58} \approx 18$.

Best possible algorithm?

We only know the upper bound $n^{\log_2 3}$.

There are (for large n) practically relevant algorithms that are faster.
The best upper bound is not known.

Lower bound: $n/2$ (each digit has to be considered at at least once)

1.6 Finde den Star

Is this constructive?

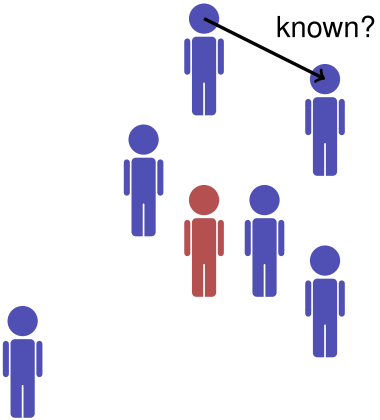
Exercise: find a faster multiplication algorithm.
Unsystematic search for a solution $\Rightarrow$ ☹.

Let us consider a more constructive example.

Example 3: find the star!

Room with $n > 1$ people.

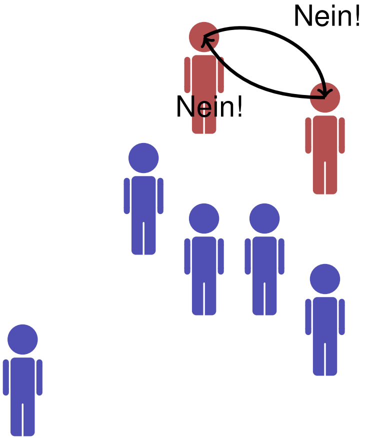
- *Star:* Person that does not know anyone but is known by everyone.
- *Fundamental operation:* Only allowed question to a person A: "Do you know B?" ($B \neq A$)



Problemeigenschaften

- Possible: no star present
- Possible: one star present
- More than one star possible?

Assumption: two stars S_1, S_2.
S_1 knows S_2 $\Rightarrow$ S_1 no star.
S_1 does not know S_2 $\Rightarrow$ S_2 no star. $\perp$



Naive solution

Ask everyone about everyone

Result:

	1	2	3	4
1	-	yes	no	no
2	no	-	no	no
3	yes	yes	-	no
4	yes	yes	yes	-

Star is 2.

Numer operations (questions): $n \cdot (n-1)$.

Better approach?

Induction: partition the problem into smaller pieces.

- $n = 2$: Two questions suffice
- $n > 2$: Send one person out. Find the star within $n-1$ people. Then check A with $2 \cdot (n-1)$ questions.

Overal
$F(n) = 2(n-1) + F(n-1) = 2(n-1) + 2(n-2) + \cdots + 2 = n(n-1).$

No benefit. ☹

Improvement

Idea: avoid to send the star out.

- Ask an arbitrary person A if she knows B.
- If yes: A is no star.
- If no: B is no star.
- At the end 2 people remain that might contain a star. We check the potential star X with any person that is out.

Analyse

$$F(n) = \begin{cases} 2 & \text{for } n = 2, \\ 1 + F(n-1) + 2 & \text{for } n > 2. \end{cases}$$

Iterative substitution:

$$F(n) = 3 + F(n-1) = 2 \cdot 3 + F(n-2) = \cdots = 3 \cdot (n-2) + 2 = 3n - 4.$$

Proof: exercise!

Moral

With many problems an inductive or recursive pattern can be developed that is based on the piecewise simplification of the problem. Next example in the next lecture.