12. Recursion

Mathematical Recursion, Termination, Call Stack, Examples, Recursion vs. Iteration, Lindenmayer Systems

Educational Objectives

- You understand how a solution to a recursive problem can be implemented in Java.
- You understand how methods are being executed in an **execution stack**.

Mathematical Recursion

- Many mathematical functions can be naturally defined recursively.
- The means, the function appears in its own definition

$$n! = \begin{cases} 1, & \text{if } n \le 1 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

Recursion in Java:

$$n! = \begin{cases} 1, & \text{if } n \le 1 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

```
// POST: return value is n!
public static int fac (int n) {
 if (n <= 1)
    return 1;
  else
    return n * fac (n-1);
}
```

Infinite Recursion

- is as bad as an infinite loop...
- ...but even worse: it burns time **and** memory

```
private static void f() {
  f(); // f() -> f() -> ... stack overflow
}
```


Recursive Functions: Termination

As with loops we need

- progress towards termination

fac(n):
terminates immediately for $n \le 1$, otherwise the function is called recusively with $< n$.

„n is getting smaller with each call."

Recursive Functions: Evaluation

Example: `fac(4)`

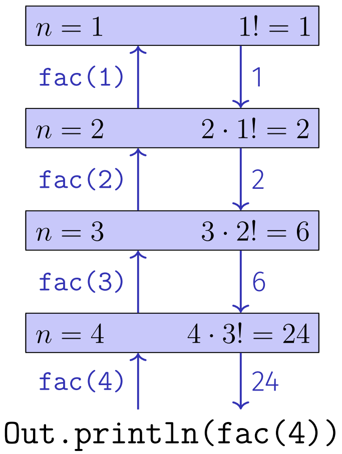
```
// POST: return value is n!
public static int fac (int n) {

  if (n <= 1) return 1;
  return n * fac(n-1); // n > 1
}
```

Initialization of the formal argument: $n = 4$
recursive call with argument $n - 1 == 3$

The Call Stack

For each method call:

- push value of the actual parameter on the stack
- work with the upper most value
- at the end of the call the upper most value is removed from the stack



Euclidean Algorithm

- finds the greatest common divisor $\gcd(a,b)$ of two natural numbers a and b
- is based on the following mathematical recursion:

$$\gcd(a,b) = \begin{cases} a, & \text{falls } b = 0 \\ \gcd(b, a \bmod b), & \text{andernfalls} \end{cases}$$

Euclidean Algorithm in Java

$$\gcd(a,b) = \begin{cases} a, & \text{falls } b = 0 \\ \gcd(b, a \bmod b), & \text{andernfalls} \end{cases}$$

```
public static int gcd (int a, int b) {
if (b == 0)
    return a;
  else
    return gcd (b, a % b);
}
```

Termination: $a \bmod b < b$, thus b is decreased for each recursive call.

Fibonacci Numbers

$$F_n := \begin{cases} 0, & \text{falls } n = 0 \\ 1, & \text{falls } n = 1 \\ F_{n-1} + F_{n-2}, & \text{falls } n > 1 \end{cases}$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

Fibonacci Numbers in Java

Laufzeit

`fib(50)` takes "forever" because it computes F_{48} two times, F_{47} 3 times, F_{46} 5 times, F_{45} 8 times, F_{44} 13 times, F_{43} 21 times ... F_1 ca. 10^9 times (!)

```
public static int fib (int n) {
  if (n == 0) return 0;
  if (n == 1) return 1;
  return fib (n-1) + fib (n-2); // n > 1
}
```

Korrektheit und Terminierung sind klar.

Fast Fibonacci Numbers

Idea:

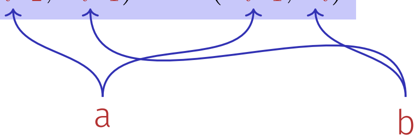
- Compute each Fibonacci number only once, in the order $F_0, F_1, F_2, \ldots, F_n$!
- Memorize the most recent two numbers (variables `a` and `b`)!
- Compute the next number as a sum of `a` and `b`!

Fast Fibonacci Numbers in Java

```
public static int fib (int n){
  if (n == 0) return 0;
  if (n <= 2) return 1;
  int a = 1;  // F_1
  int b = 1;  // F_2
  for (int i = 3; i <= n; ++i){
    int a_old = a;  // F_i-2
    a = b;                  // F_i-1
    b += a_old;             // F_i-1 += F_i-2 -> F_i
  }
  return b;
}
```

very fast, also for `fib(50)`

$$(F_{i-2}, F_{i-1}) \longrightarrow (F_{i-1}, F_i)$$



Recursion and Iteration

Recursion can *always* be simulated by

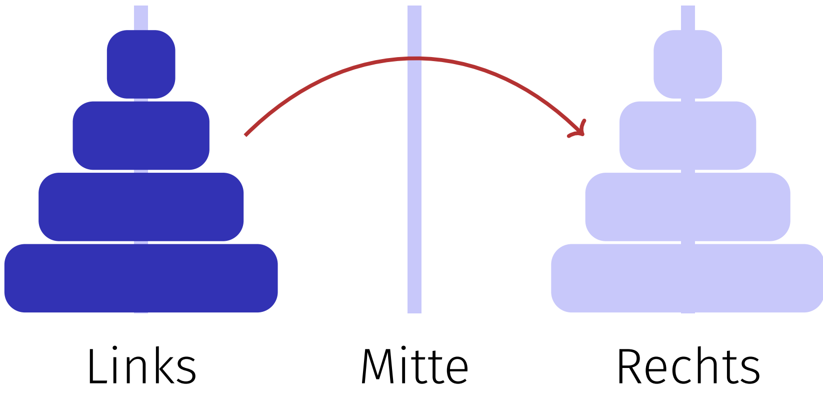
- Iteration (loops)
- explicit "call stack" (e.g. array)

Often recursive formulations are simpler, sometimes they are less efficient

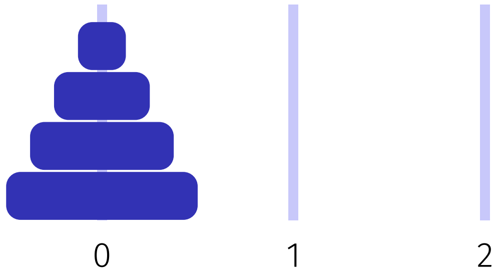
The Power of Recursion

- Some problems appear to be hard to solve without recursion. With recursion they become significantly simpler.
- Examples: *The towers of Hanoi*, The n-Queens-Problem, Sudoku-Solver, Expression Parsers, Reversing In- or Output, Searching in Trees, Divide-And-Conquer (e.g. sorting) $\rightarrow$ Informatik II,

Experiment: The Towers of Hanoi



The Towers of Hanoi – Code



Move 4 discs vom 0 to 2 with auxiliary staple 1:

```
move(4, 0, 1, 2);
```


The Towers of Hanoi – Code

```
move(4, 0, 1, 2);
```

==

1. Move 3 discs from 0 to 1 with auxiliary staple 2:
   `move(3, 0, 2, 1);`
2. Move 1 disc from 0 to 2
   `move(1, 0, 1, 2);`
3. Move 3 discs from 1 to 2 with auxiliary staple 0
   `move(3, 1, 0, 2);`

The Towers of Hanoi – Code

```
public static void move(int n, int source, int aux, int
  if (n==1){
    Out.println("move " + source + "->" + dest);
  } else {
    move(n-1, source, dest, aux);
    move(1, source, aux, dest);
    move(n-1, aux, source, dest);
  }
}
```